Typeset in JHEP style

Classical Noether's Theorem

Douglas H. Laurence

Department of Chemistry & Physics, Nova Southeastern University, Ft. Lauderdale, FL 33314

Abstract: These notes present a proof of the classical Noether's theorem, which states that for every symmetry of the Hamiltonian generated by a time-independent function, there exists a conserved quantity which is that same generator. The proof given here follows the proof given in "Classical Mechanics" by Kibble (2004).

Contents

1 Introduction

Noether's theorem is one of the most important theorems in the study of fundamental physics. While some of its most interesting applications are in the study of quantum mechanics, specifically quantum field theory, it still has important applications in classical mechanics. The three classic conservation laws in classical physics (that of linear momentum, angular momentum, and total mechanical energy) are all explained as consequences of Noether's theorem, though they are typically taught as consequences of Lagrangian or Hamiltonian mechanics[1].

Noether's theorem was proven by Emmy Noether (1882 – 1935) first in 1915 and then published in 1918. The statement of it that is most widely known is the more general form,

> For every continuous symmetry of the action of a system, that system has a corresponding conservation law.

This is a much stronger version of the form that I will show in this paper, which is easily proven using Hamiltonian mechanics, and only applies in the context of classical physics[2].

[1] This is the main reason why Noether's theorem isn't considered "essential" in the study of classical mechanics.

[2] This is because it relies on the use of Poisson brackets which need to be extended to commutator brackets in quantum mechanics.

2 A Quick Review of Hamiltonian Mechanics

For a classical system, the Lagrangian is defined as

$$L(q_\alpha, \dot{q}_\alpha) = T - V \tag{2.1}$$

where T is the system's kinetic energy, and V is the system's potential energy, the q_α's are the canonical coordinates, and the $\dot{q}_\alpha$'s are the canonical velocities. From the Lagrangian, the canonical momenta are defined as

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \tag{2.2}$$

The Hamiltonian is defined as the Legendre transformation of the Hamiltonian. A Legendre transformation is the transformation of a function $f(A)$, where A is a set of variables, to a function $Df(B)$, where B is a set of *conjugate* variables. In the case of the Lagrangian-to-Hamiltonian transformation, $A = \{q_\alpha, \dot{q}_\alpha\}$, and $B = \{q_\alpha, p_\alpha\}$. The canonical momenta are sometimes referred to as the *conjugate* momenta for this reason. The Hamiltonian, defined as the Legendre transformation of the Lagrangian, is given by

$$H(q_\alpha, p_\alpha) = \sum_\alpha p_\alpha \dot{q}_\alpha - L(q_\alpha, p_\alpha) \tag{2.3}$$

Note that if the following two conditions are met, then the Hamiltonian is equal to the total mechanical energy:

1. The kinetic energy is of the form

$$T = \sum_\beta c_\beta \left(\dot{q}_\beta\right)^2$$

2. The potential energy is not a function of velocity, but only position

$$V \neq V\left(\dot{q}_\alpha\right)$$

Then, the canonical momenta are given by:

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} = 2c_\alpha \dot{q}_\alpha - \underbrace{\frac{\partial V}{\partial} \dot{q}_\alpha}_{=0} = 2c_\alpha \dot{q}_\alpha$$

So the Hamiltonian is given by

$$H = \sum_\alpha \left(2c_\alpha \dot{q}_\alpha\right) \dot{q}_\alpha - \left[\sum_\alpha c_\alpha \left(\dot{q}_\alpha\right)^2 - V\right] = \sum_\alpha c_\alpha \left(\dot{q}_\alpha\right)^2 + V = T + V = E$$

Along with the Hamiltonian, we have Hamilton's equations, the analogue to the Euler-Lagrange equations of Lagrangian mechanics. They are:

$$\frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha \tag{2.4a}$$

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha \tag{2.4b}$$

3 Transformations of a System

From equation (2.4a), we know that if a system is described by a Hamiltonian that is independent of a canonical coordinate q_α, then the canonical momentum p_α of that system is conserved. That is, if

$$\frac{\partial H}{\partial q_\alpha} = 0$$

i.e. the Hamiltonian has no explicit dependence on q_α, then

$$\dot{p}_\alpha \equiv 0$$

i.e. that the canonical momentum p_α doesn't change with time, and so is a constant. Many systems can have their Hamiltonians expressed in such a way as to make the conserved quantity obvious. But what if the Hamiltonian is expressed in a way that doesn't make it obvious?

Consider, for example, the Hamiltonian associated with a central force:

$$H = \frac{p_x^2 + p_y^2}{2m} + V\left(\sqrt{x^2 + y^2}\right) \tag{3.1}$$

I purposefully expressed this Hamiltonian in Cartesian coordinates to make the problem more difficult. When expressed in polar coordinates, the Hamiltonian will be explicitly independent of the angular coordinate ϕ, and thus the canonical momentum p_ϕ – that is, the angular momentum – will be conserved. But this is only because I know beforehand that the angular momentum of a rotationally invariant system is conserved, and this is easily seen if I express the Hamiltonian in polar coordinates. What if this wasn't known beforehand, and I just wrote the Hamiltonian down in the most convenient coordinates?

As written, the Hamiltonian is dependent on both canonical coordinates, x and y, so there is no apparent conserved momentum. There must be some method that we can develop in order to find these conserved quantities regardless of how we choose to express the Hamiltonian. Because we know that a system under the influence of a central force is rotationally invariant, we can make a "guess" and investigate a rotation and see if this leads to some general method for finding these conserved quantities.

The most convenient way to express a rotation is through the use of the rotation matrix. For a counterclockwise rotation about the z-axis (i.e. a rotation of the xy-plane) by a very small angle $\delta\phi$, the rotation matrix is

$$R(\delta\phi) = \begin{pmatrix} \cos\delta\phi & -\sin\delta\phi \\ \sin\delta\phi & \cos\delta\phi \end{pmatrix} \tag{3.2}$$

Using the small angle approximations, $\sin\delta\phi \approx \delta\phi$ and $\cos\delta\phi \approx 1$, the rotation matrix becomes:

$$R(\delta\phi) \approx \begin{pmatrix} 1 & -\delta\phi \\ \delta\phi & 1 \end{pmatrix} \tag{3.3}$$

Now we want to rotate a position vector, $\mathbf{x} = (x\ \ y)^T$, and a momentum vector, $\mathbf{p} = (p_x\ \ p_y)^T$. We actually only need to rotate one; the results will be identical for both. Rotating the position vector:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R(\delta\phi)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -\delta\phi \\ \delta\phi & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y\delta\phi \\ x\delta\phi + y \end{pmatrix}$$

If we say $x' = x + \delta x$ and $y' = y + \delta y$, then

$$\delta x = -y\delta\phi \tag{3.4a}$$

$$\delta y = x\delta\phi \tag{3.4b}$$

Likewise, we'll also have

$$\delta p_x = -p_y\delta\phi \tag{3.5a}$$

$$\delta p_y = p_x\delta\phi \tag{3.5b}$$

The set of equations $\{\delta q_\alpha, \delta p_\alpha\}$ is what defines our transformation. With our transformation defined, we can show that the Hamiltonian is invariant (under this transformation). This is easiest to show by treating δ as an infinitesimal operator and taking the implicit derivative of x^2+y^2 and $p_x^2+p_y^2$. Once again, the results will be the same for both, so I will only perform this for one:

$$\delta(x^2 + y^2) = \delta(x^2) + \delta(y^2) = 2x\delta x + 2y\delta y = 2x(-y\delta\phi) + 2x(x\delta\phi) = 0$$

So, the quantities $x^2 + y^2$ and $p_x^2 + p_y^2$ are both invariant under the rotation. Since the Hamiltonian for this system, given by equation (3.1), depends only upon $p_x^2+p_y^2$ and $\sqrt{x^2 + y^2}$, it, too, is invariant under this transformation. So, we've shown that the Hamiltonian is invariant, but what is the conserved quantity that goes along with this transformation?

4 Generators of a Transformation

Before answering this question, it is worth introducing a new concept. I want to define a function $G(q_\alpha, p_\alpha, t)$, referred to as a **generator** or generating function, such that the transformation is defined as

$$q_\alpha = \frac{\partial G}{\partial p_\alpha}\delta\lambda \tag{4.1a}$$

$$p_\alpha = -\frac{\partial G}{\partial q_\alpha}\delta\lambda \tag{4.1b}$$

In this sense, we would say that G *generates* the transformations[3]. $\delta\lambda$ is known as the **parameter** of the transformation, and is an arbitrary infinitesimal.

For the example we have been considering, let's try the generating function

$$G = xp_y - yp_x \tag{4.2}$$

To see if this is the generator of the transformations, we just need to apply equations (4.1a) and](4.1b):

$$x = \frac{\partial(xp_y - yp_x)}{\partial p_x}\delta\lambda = -y\delta\lambda$$

$$y = \frac{\partial(xp_y - yp_x)}{\partial p_y}\delta\lambda = x\delta\lambda$$

$$p_x = \frac{\partial(xp_y - yp_x)}{\partial x}\delta\lambda = -p_y\delta\lambda$$

$$p_y = \frac{\partial(xp_y - yp_x)}{\partial y}\delta\lambda = p_x\delta\lambda$$

We can see that these are exactly our transformation equations, (3.4a) through (3.5b), if the parameter λ is just the angle ϕ. If you didn't recognize is, the generator that I chose is just the angular momentum, J.

So far, I've shown that a transformation can lead to an invariant Hamiltonian, and that a function can generate that transformation, but I still haven't connected these ideas to the conservation of a quantity associated with these transformations. In order to do this, I need to discuss Poisson brackets.

5 Poisson Brackets

Consider a function $F(q_\alpha, p_\alpha, t)$. Under the transformation given by $\{\delta q_\alpha, \delta p_\alpha\}$, the function F changes like:

[3] Mathematically, these generators are elements of a Lie algebra describing the transformations of a Lie group.

$$\delta F = \sum_\alpha \left(\frac{\partial F}{\partial q_\alpha} \delta q_\alpha + \frac{\partial F}{\partial p_\alpha} \delta p_\alpha \right) \tag{5.1}$$

Using equations (4.1a) and (4.1b), the above equation becomes:

$$\delta F = \sum_\alpha \left(\frac{\partial F}{\partial q_\alpha} \frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha} \frac{\partial G}{\partial q_\alpha} \right) \delta\lambda \tag{5.2}$$

The summation term is known as a **Poisson bracket**,

$$\{F, G\} = \sum_\alpha \left(\frac{\partial F}{\partial q_\alpha} \frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha} \frac{\partial G}{\partial q_\alpha} \right) \tag{5.3}$$

Using the Poisson bracket, we can write the change in F simply as

$$\delta F = \{F, G\} \delta\lambda \tag{5.4}$$

Notice that, from the definition, the Poisson bracket has an anticommutativity property:

$$\{F, G\} = -\{G, F\} \tag{5.5}$$

This can be seen by simply exchanging the functions F and G in equation (5.3). You need to reverse the order of the terms so that the q_α derivative of the first function, now G, appears as the first term. This induces a negative sign.

If we want to consider the time derivative of the function F, we need to use the chain rule:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_\alpha \left(\frac{\partial F}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial F}{\partial p_\alpha} \dot{p}_\alpha \right)$$

Using Hamilton's equations, given by equations (2.4a) and (2.4b), the above equation becomes:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_\alpha \left(\frac{\partial F}{\partial q_\alpha} \frac{\partial H}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha} \frac{\partial H}{\partial q_\alpha} \right)$$

Or, in terms of the Poisson bracket $\{F, H\}$[4],

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\} \tag{5.6}$$

[4]This equation is the classical analogue to Heisenberg's equation, one of the dynamical equations in quantum mechanics. The Heisenberg equation is equivalent to Schrödinger's equation, simply presenting a different way of looking at quantum mechanics.

6 The Classical Noether's Theorem

Now we are ready to put everything together. Suppose that we have a transformation, described by $\{\delta q_\alpha, \delta p_\alpha\}$, generated by some function G which does not *explicitly* depend on time, under which the Hamiltonian is invariant. We can say, using equation (5.4), that

$$\delta H = \{H, G\}\delta\lambda = 0$$

By the anticommutativity property of the Poisson brackets, this means that $\{G, H\} = 0$ as well. Since we have already stipulated that G not explicitly depend on time, we know that $\partial G/\partial t = 0$, so by equation (5.6), we see that

$$\frac{dG}{dt} = \{G, H\} \equiv 0 \tag{6.1}$$

So, if a transformation generated some $G \neq G(t)$ leaves the Hamiltonian invariant, then the generator itself is conserved. **This is the classical version of Noether's theorem.**

An important thing to note about this: just because G does not explicitly depend on time doesn't mean that it's guaranteed that it doesn't *change* with time. Take the generator given in our example: the angular momentum, which was given by

$$J = xp_y - yp_x$$

This generator doesn't explicitly depend on time, but the variables it does depend on – x, y, p_x, and p_y – all definitely depend on time. So there's no reason to automatically assume that $dJ/dt = 0$. The only reason this is true is because the angular momentum generates a transformation that leaves the Hamiltonian invariant, and thus is a conserved quantity, by Noether's theorem.