TYPESET IN JHEP STYLE

Kepler's Laws of Planetary Motion

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ABSTRACT: This paper will give detailed derivations of Kepler's three laws of planetary motion based on first principles in classical mechanics. All that will be assumed is that a central force conserves angular momentum and that a conservative force conserves total mechanical energy (both of which are easily provable facts). Gravity, being both a central and a conservative force, conserves both total mechanical energy and angular momentum. Kepler's first law is a pretty long, pretty intense derivation, but after that, Kepler's second and third laws are very short and fairly simple.

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1 Kepler's First Law

> *All satellites orbits are elliptical with the attracting body at one of the foci of the ellipse.*

You can derive Kepler's first law in a couple of ways, but ultimaty you're going to have to either use something about the force of gravity (e.g. use Newton's laws) or something about the energy and momentum of a satellite under the influence of gravity (e.g. use energy and momentum conservation) in order to derive it. Here, I'll present the derivation of Kepler's first law using energy and angular momentum conservation. Since gravity is a central, conservative force, it conserves both angular momentum and energy. First, let's write down the energy and angular momentum of a particle moving around some gravitational source.

For orbital motion in a plane, we have the following equaitons

$$E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r) \tag{1.1}$$

$$J = mr^2\dot{\theta} \tag{1.2}$$

We want to rewrite equation (1.1) using equation (1.2):

$$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U(r) \tag{1.3}$$

Let's define a new variable w

$$w = \frac{1}{r}$$

so that

$$\frac{dw}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} \tag{1.4}$$

Using the chain rule, we know

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta}$$

Now, plugging in equation (1.4) and using equation (1.2), we get

$$\dot{r} = -r^2\dot{\theta}\frac{dw}{d\theta} = -\frac{J}{m}\frac{dw}{d\theta} \tag{1.5}$$

And, plugging this result into equation (1.5), we get

$$E = \frac{J^2}{2m}\left(\frac{dw}{d\theta}\right)^2 + \frac{J^2}{2m}w^2 + U(w) \tag{1.6}$$

Since we've replaced the variable r in our energy equation with w, we need to rewrite our potential energy in terms of w. Let's consider a potential energy of the form

$$U(r) = -\frac{k}{r} \tag{1.7}$$

where $k > 0$. In the case of gravity, we know

$$k = GMm \tag{1.8}$$

where G is Newton's gravitational constnat, M is the mass of the attracting body, and m is the mass of the satellite. Rewriting equation (1.7) in terms of w,

$$U(w) = -kw$$

And, plugging this into equation (1.6), we see the energy is

$$E = \frac{J^2}{2m}\left(\frac{dw}{d\theta}\right)^2 + \frac{J^2}{2m}w^2 - kw$$

Now we want to define a constant, l, such that

$$l = \frac{J^2}{mk} \tag{1.9}$$

Note that since l depends on J, and as a central force, gravity conserves J, l is indeed a constant. If we plug this into equation (1.6), and multiply by $2/k$, we see that

$$l\left(\frac{dw}{d\theta}\right)^2 + lw^2 - 2w = \frac{2E}{k} \tag{1.10}$$

Multiply by l and adding 1 to each side, equation (1.10) becomes

$$l^2\left(\frac{dw}{d\theta}\right)^2 + l^2w^2 - 2wl + 1 = \frac{2El}{k} + 1 \tag{1.11}$$

Noting that

$$l^2w^2 - 2wl + 1 = (lw - 1)^2$$

we want to define a third variable,

$$z = lw - 1 \tag{1.12}$$

to plug into equation (1.11). Before continuing, recall that l is a constant. This means that

$$\frac{dz}{d\theta} = l\frac{dw}{d\theta}$$

since l is a constant. Then, equation (1.11) becomes

$$\left(\frac{dz}{d\theta}\right)^2 + z^2 = \frac{2El}{k} + 1 \tag{1.13}$$

We now want to define a dimensionless constant e such that

$$e^2 = \frac{2El}{k} + 1 \tag{1.14}$$

Plugging this in, equation (1.13) becomes

$$e^2 = \left(\frac{dz}{d\theta}\right)^2 + z^2 \tag{1.15}$$

This is a differential equation in $z(\theta)$. Note that e^2 depends upon E. Since gravity is a conservative force, E should be a constant, so e is a constant as well.

Consider a possible solution to equation (1.15):

$$z = e\cos\theta \tag{1.16}$$

Let's check to see if this does, in fact, solve the differential equation. Noting that

$$\frac{dz}{d\theta} = -e\sin\theta$$

since e is a constant. Plugging this into the right-hand-side of equation (1.15), we see that

$$\left(\frac{dz}{d\theta}\right)^2 + z^2 = e^2\sin^2\theta + e^2\cos^2\theta = e^2$$

So, indeed, the solution presented in equation (1.16) does solve our differential equation. Recalling the definition of z, given by equation (1.12), our solution presented above becomes

$$lw - 1 = e\cos\theta \tag{1.17}$$

or, recalling that $r = w^{-1}$,

$$r = \frac{l}{e\cos\theta + 1} \tag{1.18}$$

This is an orbit equation in polar coordinates for an ellipse. However, I find it more instructive to continue working the equation to write it in Cartesian coordinates. We'll gain insight into what the semi-major and semi-minor axes will be of the ellipse if we continue along this route. Manipulating equation (1.18), we see that

$$\begin{aligned} & r(e\cos\theta + 1) = e(r\cos\theta) + r = l \\ \Rightarrow\ & r = l - e(r\cos\theta) \end{aligned}$$

So, squaring both sides,

$$r^2 = (l - e(r\cos\theta))^2 \tag{1.19}$$

Now, we want to use the coordinate transformations from polar to Cartesian in a plane

$$\begin{aligned} & r^2 = x^2 + y^2 \\ & r\cos\theta = x \end{aligned}$$

Plugging these in, equation (1.19) becomes

$$x^2 + y^2 = (l - ex)^2 \tag{1.20}$$

Now, we want to rearrange the above equation to group all the x's and y's together:

$$(1 - e^2)x^2 + 2lex + y^2 = l^2$$

Completing the square on the left-hand-side of the above equation,

$$\begin{aligned} & x^2 + \frac{2lex}{1-e^2} + \frac{y^2}{1-e^2} = \frac{l^2}{1-e^2} \\ \Rightarrow\ & x^2 + \frac{2lex}{1-e^2} + \left(\frac{le}{1-e^2}\right)^2 - \left(\frac{le}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{l^2}{1-e^2} \end{aligned}$$

So,

$$\left(x + \frac{lex}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{l^2}{1-e^2} + \left(\frac{le}{1-e^2}\right)^2 \tag{1.21}$$

Notice that the right-hand-side of the above equation is equal to

$$\frac{l^2}{1-e^2} + \left(\frac{le}{1-e^2}\right)^2 = \frac{l^2(1-e^2)}{(1-e^2)^2} + \frac{l^2e^2}{(1-e^2)^2} = \frac{l^2}{(1-e^2)^2}$$

Now we can define another constant,

$$a = \frac{l}{1 - e^2} \tag{1.22}$$

Note that since l and e are both constants, a is indeed a constant. Plugging this in, equation (1.21) becomes

$$(x + ae)^2 + \frac{y^2}{1 - e^2} = \frac{l^2}{(1 - e^2)^2} \tag{1.23}$$

Dividing both sides by a factor of a^2, we see the above equation becomes

$$\frac{(x + ae)^2}{a^2} + \frac{y^2(1 - e^2)}{l^2} = 1 \tag{1.24}$$

If we make one last substitution,

$$b = \frac{l}{\sqrt{1 - e^2}} \tag{1.25}$$

where b is a constant since, once again, both l and e are constants, then equation (1.24) becomes

$$\frac{(x + ae)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1.26}$$

which is an ellipse with a as the semi-major axis and b the semi-minor axis. By definition, the eccentricity of an ellipse is given by

$$\text{eccentricity} = \sqrt{1 - \frac{b^2}{a^2}}$$

Plugging in our substitutions for a and b,

$$\text{eccentricity} = \sqrt{1 - \frac{\frac{l^2}{1-e^2}}{\frac{l^2}{(1-e^2)^2}}} = \sqrt{1 - (1 - e^2)} = e$$

So our constant e is the eccentricity of the ellipse. Note, then, that the coordinate x is shifted by ae. By definition of the eccentricity, ae is always the distance from the center of an ellipse to a focus (called the "linear eccentricity"). This means that the attracting body, which lies at $x = 0$, $y = 0$ (i.e. at $r = 0$), lies at a focus of the ellipse, not at the center. This is exactly what Kepler's first law says.

2 Kepler's Second Law

The rate at which a line sweeps area around a satellite's orbit is a constant.

By definition, the angular momentum is

$$J = mr^2\dot{\theta} \tag{2.1}$$

and is a constant. For a small sweep of a line along a satellite's orbit, the shape of the area covered is roughly triangular. The radius of the orbit is r, by definition, and the arc length swept is $rd\theta$. Since the area of a triangle is one-half base times height, we can say that the infinitesimal area swept is

$$dA = \frac{1}{2}(r)(rd\theta) = \frac{1}{2}r^2 d\theta \tag{2.2}$$

Or, that the rate at which area is being swept is

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{J}{2m} \tag{2.3}$$

which is a constant, thus proving Kepler's second law.

3 Kepler's Third Law

The period, squared, of a satellite's orbit is proportional to the orbit's semi-major axis, cubed.

Using Kepler's second law, and the fact that the area of an ellipse is πab, we see that

$$dt = \frac{2m}{J}dA \quad \Rightarrow \quad T = \frac{2\pi mab}{J}$$

where T is the period. The above relation is so simple to find because dA/dt is a constant.

Let's go back to our derivation of the first law and borrow equation (1.22):

$$a = \frac{l}{1-e^2}$$

Borrowing equation (1.25) from the first section as well, we see that b is related to a:

$$b = \frac{l}{\sqrt{1-e^2}} \quad \Rightarrow \quad b^2 = \frac{l^2}{1-e^2} = al$$

Recalling the definition of l, given by equation (1.9), we see that

$$J^2 = mkl$$

So, using our above equation for the period T, we see that

$$T^2 = \frac{4\pi^2 m^2 a^2 b^2}{J^2}$$

Plugging in our results for b^2 and J^2 given above:

$$T^2 = \frac{4\pi^2 m^2 a^2 (al)}{mkl} = \frac{4\pi^2 m a^3}{k}$$

Recalling the definition of k from the first section, given by equation (1.8),

$$k = GMm$$

we see that the m's cancel from the numerator and the denominator, resulting in the equation for Kepler's third law:

$$T^2 = \frac{4\pi^2}{GM} a^3 \tag{3.1}$$

Note that if you were to perform an analysis on circular motion using the equation for centripetal acceleration and Newton's law of universal gravitation, you would get the same equation for Kepler's third law, however it would be in terms of the radius of the circular orbit instead of the semi-major axis of an elliptical orbit.