Friedmann Cosmology

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1 Introduction

Friedmann cosmology is a catch-all term for cosmology centered around the Friedmann equations, which are solutions to Einstein's field equation for the Robertson-Walker metric. The Robertson-Walker metric is, of course, the most general metric for a spacetime satisfying the cosmological principle, i.e. that it be homogeneous and isotropic in space. For a derivation of the Robertson-Walker metric, see my note about it. Once the Friedmann equations are derived from the Einstein field equation, we can explore the implications that they have on the cosmology of our universe, whether the universe is flat, spherical or hyperbolic. We'll then be able to see what implications current cosmological parameters have on the cosmology of our universe, especially on the geometry of it.

The Friedmann equations were first derived by Alexander Friedmann in 1922[1], which gave a theoretical explanation for Hubble's law even before there was a Hubble's law. Hubble's law wasn't discovered until 1929, so of course it wasn't called this in Friedmann's paper, but he did present a solution for an expanding universe as a possibility, and gave an equation for the velocity of the expansion. In fact, Einstein had already considered the possibility of an expanding universe, but rejected it as unphysical. He then "butchered" (as far as he was concerned) his own field equation in 1917[2] to include something now termed the cosmological constant Λ to prevent such an expansion. After it was experimentally confirmed by Hubble, Einstein was very disappointed that he didn't stick by his equation in it's original form, because he would have predicted the expansion of the universe *a priori*. (This is distinct from what Friedmann did, which was present an expanding unvierse as a possible solution. Einstein also would have done this a few years before Friedmann

[1] A. Friedmann, "Über die Krümmung des Raumes," *Z. Phys.*, **10**, 377 (1922).

[2] A. Einstein, "Kosmologische Betrachtungen zur allgemeinen Relativitaetstheorie," *Sitzber. K. Preuss. Akad. Wiss.*, **1**, 142 (1917).

published his 1922 paper.) Of course, it turns out now that to explain dark energy, we might need the cosmological constant. But more on this later.

There are many important parameters in cosmology, but the four most important to us are going to be the (relative) matter density, Ω_m, the (relative) dark energy density, Ω_Λ, the total (relative) energy density, Ω, and the (current) Hubble constant, H_0. While the first three are known very accurately, there is a lot of difficulty in measuring the Hubble constant, so accepted numbers vary from 65 km s^{-1}/Mpc to 75 km s^{-1}/Mpc, with the Planck satellite giving a value of 67.31 km s^{-1}/Mpc (as of the 2015 data). The majority of the discussion on cosmology will revolve around these four quantities and how they affect various observations and the overall evolution of our universe.

2 Derivation of the Friedmann Equations

First of all, I want to point out that there are two Friedmann equations, in the sense that these were the two equations that Friedmann published in his 1922 papers and are generally given his name. However, often times a third equation is lumped into the mix, replacing one of the two Friedmann equations. This third equation is lumped in because it isn't independent of the two Friedmann equations; a linear combination of the two will give this third equation. But, technically, the third equation is a consequence of conservation of energy, and is typically referred to as the continuity equation. I will make the distinction between these three, though I will use a mixture of the three of them interchangeably depending on which equation gives a solution to a problem more easily; this is what you should always do in physics, anyways.

We start our derivation with the Robertson-Walker metric, given in terms of arbitrary (normalized) curvature k:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{1}$$

Note that $k = 1$ corresponds to spherical geometry, $k = 0$ corresponds to flat geometry, and $k = -1$ corresponds to hyperbolic geometry. Also note that, in this form, I am keeping the units of distance in my coordinate r, while keeping the scale-factor $a(t)$ unitless. In this picture, the scale factor will simply tell me how 1m changes from how it's defined now, such that $a(t = 0) = 1$, to any time in the past ($t < 0$) or any time in the future ($t > 0$).

The first thing we need to do is to compute all non-zero Christoffel symbols. These are, excluding any that can be found with the symmetry $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$:

$$\begin{aligned}
&\Gamma^t_{rr} = \frac{a\dot{a}}{1-kr^2} && \Gamma^t_{\theta\theta} = a\dot{a}r^2 && \Gamma^t_{\phi\phi} = a\dot{a}r^2\sin^2\theta \\
&\Gamma^r_{rr} = \frac{kr}{1-kr^2} && \Gamma^r_{rt} = \frac{\dot{a}}{a} && \Gamma^r_{\theta\theta} = -r(1-kr^2) \\
&\Gamma^r_{\phi\phi} = -r(1-kr^2)\sin^2\theta && \Gamma^\theta_{t\theta} = \frac{\dot{a}}{a} && \Gamma^\theta_{r\theta} = \frac{1}{r} \\
&\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta && \Gamma^\phi_{t\phi} = \frac{\dot{a}}{a} && \Gamma^\phi_{r\phi} = \frac{1}{r} \\
&\Gamma^\phi_{\theta\phi} = \cot\theta
\end{aligned} \tag{2}$$

A better way to get to the Einstein equations is to compute the Ricci tensor direction, bypassing the Riemann curvature tensor, which will save us a bunch of calculations. To compute the Ricci tensor directly, note the following equation:

$$R_{\mu\nu} = \frac{\partial}{\partial x^{\nu}}\Gamma_{\mu} - \frac{\partial}{\partial x^{\rho}}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\nu}\Gamma_{\rho} + \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} \tag{3}$$

where I have defined a "reduced" Christoffel symbol:

$$\Gamma_{\mu} = \Gamma^{\nu}_{\mu\nu} \tag{4}$$

just to make the calculations easier. The non-zero reduced Christoffel symbols will be:

$$\begin{aligned} &\Gamma_t = \frac{3\dot{a}}{a} \qquad \Gamma_r = \frac{2-kr^2}{r(1-kr^2)} \qquad \Gamma_{\theta} = \cot\theta \\ &\Gamma_{\phi} = 0 \end{aligned} \tag{5}$$

So, the four non-zero components of the Ricci tensor are:

$$\begin{aligned} R_{tt} &= -\frac{3\ddot{a}}{a} & R_{rr} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2} \\ R_{\theta\theta} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k) & R_{\phi\phi} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k)\sin^2\theta \end{aligned} \tag{6}$$

Finally, the Ricci scalar, R, is just the trace of the Ricci tensor $R = R^{\mu}_{\mu}$:

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] \tag{7}$$

Now that we know all the relevant geometric objects, we can compute the stress-energy tensor for our universe, which is the second piece of information (after the geometric piece) needed to solve the Einstein field equation. In cosmology, we always use the stress-energy tensor of a perfect fluid, because we consider only three sources of energy: matter, treated like a fluid of particles at rest, radiation, treated like a fluid of ultra-relativistic particles, and dark energy, which is treated as a fluid with extremely strange properties. The stress energy tensor, represented as a 4×4 matrix $||T_{\mu\nu}||$, for a perfect fluid is:

$$||T_{\mu\nu}|| = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & ||\gamma_{ij}p|| & \\ 0 & & & \end{pmatrix} \tag{8}$$

where ρ is the energy density (or mass density, as I always use the units $c = 1$), p is the pressure produced by the fluid, and γ_{ij} is the 3-space metric, given by the 3-space Robertson-Walker equation:

$$a^2(t)d\sigma^2 = a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{9}$$

Often times it's more useful to know the stress-energy tensor with mixed contravariant and covariant indices. In this case, we just need to raise an index on $T_{\mu\nu}$, like $T^{\mu}_{\nu} = g^{\mu\rho}T_{\rho\nu}$, which will add a negative sign on the ρ term due to g^{00}, but free the p terms due to $\gamma^{ij}\gamma_{jk} = \mathbb{I}_3$ (where $\mathbb{I}_3$ is the 3×3 identity matrix). So, writing the matrix as a diagonal, we have:

$$||T^{\mu}_{\nu}|| = \text{diag}(-\rho, p, p, p) \tag{10}$$

This is particularly convenient because it's very easy to compute the trace:

$$\text{tr}(\mathbf{T}) = T^{\mu}_{\mu} = -\rho + 3p \tag{11}$$

where $\mathbf{T}$ is the symbol I'm using for the matrix representation of the stress energy tensor with mixed indices.

Something interesting, but maybe expected, is that the three perfect fluids each have an equation of state, much like the ideal gas law, which relates energy density ρ to pressure p, in the form:

$$p = w\rho \tag{12}$$

where w is a constant that depends solely on the type of fluid being considered: matter has $w = 0$, radiation has $w = 1/3$, and dark energy has $w = -1$. The derivations for matter and radiation I have included as an appendix, but I will justify the claim that $w = -1$ for dark energy latter on in this note when we cover dark energy in depth.

As I mentioned at the start of this section, there are two equations typically referred to as the Friedmann equations, with a third equation typically referred to as the continuity equation because it's derived from conservation of energy, even though the three equations are not independent of one another. We will derive the continuity equation first because we already have the stress-energy tensor, so the condition for conservation of energy is easy to solve:

$$\nabla_{\mu} T^{\mu}_{0} = 0 \;\;\Rightarrow\;\; \frac{\partial}{\partial x^{\mu}} T^{\mu}_{0} + \Gamma^{\mu}_{\mu\lambda} T^{\lambda}_{0} - \Gamma^{\lambda}_{\mu 0} T^{\mu}_{\lambda} = 0 \;\;\Rightarrow\;\; -\frac{\partial \rho}{\partial t} - \frac{3\dot{a}}{a}(\rho + p) = 0$$

So, the continuity equation is:

$$\frac{\partial \rho}{\partial t} = -\frac{3\dot{a}}{a}(\rho + p) \tag{13}$$

If we plug in our equation of state, we can get a more useful form of the continuity equation:

$$\frac{\dot{\rho}}{\rho} = -3(1 + w)\frac{\dot{a}}{a} \tag{14}$$

The continuity equation, in the above form, is fairly simple to solve by integration:

$$\rho = C a^{-3(1+w)} \tag{15}$$

where C is an integration constant (or a proportionality constant, however you want to think about it). What does this result tell us? Well, for matter, $w = 0$, so the energy density scales like:

$$\rho_m \propto a^{-3} \tag{16}$$

Does this result make sense? If we switch from our view of the scale-factor as changing the value of 1m to our view in terms of comoving coordinates, where $a(t)$ carries the physical units, then clearly what the above result is saying is that matter density depends on a^{-3}. This absolutely makes sense, since matter density should be proportional to 1/volume.

What about for radiation? Our solution of the continuity equation tells us, for $w = 1/3$, that:

$$\rho_r \propto a^{-4} \tag{17}$$

Once again, let's check whether this makes sense. The energy density of radiation should be something like $h\nu$/volume. But $\nu = 1/\lambda$ (for $c = 1$), so the energy density of radiation should be something like $h/\lambda *$volume. Well, as the scale factor increases, volume increases as a^3 and the wavelength is stretched as a, meaning that ρ_r should absolutely go like a^{-4}.

What about for dark energy? This is our first change to take a peak at what dark energy actually is, from the assumption that $w = -1$ is correct. If it is, then for dark energy:

$$\rho_\Lambda \propto a^0 \tag{18}$$

This is a very peculiar result: the energy density of dark energy is a constant, no matter the scale of the universe. This means that as the universe increases in size, i.e. as a grows, ρ_Λ remains constant while both ρ_m and ρ_r decrease. So, while there is a fixed number of photons (or a fixed photon energy if you'd like) and a fixed amount of matter in the universe, such that both of their *densities* decrease as the size of the universe grows, it's the energy density of dark energy that's fixed, so the actual amount of dark energy continuously grows as the universe gets larger and larger. This will be a very important feature of dark energy later on, and it's due solely to the equation of state, i.e. the fact that $w = -1$ for dark energy.

Now we're ready to derive the two Friedmann equations. Recall Einstein's field equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \tag{19}$$

where I'm setting $G = 1$. Using our Ricci tensor, equation (6), our Ricci scalar, (7), and the covariant stress-energy tensor given by equation (8), the 00 term of the Einstein equation is:

$$R_{00} - \frac{1}{2} g_{00} R = 8\pi T_{00} \;\; \Rightarrow \;\; -\frac{3\ddot{a}}{a} - \frac{1}{2} * 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] (-1) = 8\pi\rho$$

$$\Rightarrow \;\; -\frac{3\ddot{a}}{a} + \frac{3\ddot{a}}{a} + 3 \left(\frac{\dot{a}}{a} \right)^2 + \frac{3k}{a^2} = 8\pi\rho$$

This gives us our first Friedmann equation:

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2} \tag{20}$$

The second Friedmann equation will be found using the 11 component of Einstein's equation:

$$R_{11} - \frac{1}{2} g_{11} R = 8\pi T_{11}$$

$$\Rightarrow \;\; \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} - \frac{1}{2} * 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] \left(\frac{a^2}{1 - kr^2} \right) = 8\pi \left(\frac{a^2}{1 - kr^2} \right) p$$

$$\Rightarrow \;\; -2a\ddot{a} - \dot{a}^2 - k = 8\pi a^2 p$$

This is in, basically, the form that Friedmann published the second equation in. However, we want to modify this equation my using the first equation, to find a simplified form of the second equation which is more commonly used. If we multiply equation (20) by $3a^2$, and move the k term over to the left-hand-side, the first Friedmann equation becomes:

$$3\dot{a}^2 + 3k = 8\pi a^2 \rho$$

If we then multiply the 11 solution by 3 and add it to the first Friedmann equation given above, which will allow the $\pm 3\dot{a}^2$ and the $\pm 3k$ terms to cancel from each equation, we come to the equation:

$$-6a\ddot{a} - 3\dot{a}^2 - 3k + 3\dot{a}^2 + 3k = 8\pi a^2(3p) + 8\pi a^2\rho \;\Rightarrow\; -6a\ddot{a} = 8\pi a^2(\rho + 3p)$$

Simplifying the above result, and re-writing the first Friedmann equation, equation (20), for convenience, the two Friedmann equations are:

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi}{3}\rho - \frac{k}{a^2} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi}{3}(\rho + 3p) \end{aligned} \tag{21}$$

Combining the two Friedmann equations with the continuity equation, given by equation (14), we have the three fundamental equations of cosmology (as I call them; I don't know if this set of equations goes by any standardized name). Any pair from the set of these three equations will completely describe the cosmology for a universe with a curvature k.

3 Cosmological Parameters

The first observable that we will define in our exploration of Friedmann cosmologies is the Hubble parameter:

$$H(t) \equiv \frac{\dot{a}}{a} \tag{22}$$

While this was originally called the Hubble constant, at least the current-time value $H(t = 0) = H_0$ was, it no longer makes any sense to call it a constant since it clearly varies with time, hence the switch to the word "parameter." As stated in the introduction, the Hubble parameter is difficult to measure, and there isn't really a consensus on what value it should be. While the range of accepted values is typically between 65 km s^{-1}/Mpc and 75 km s^{-1}/Mpc, with the Planck satellite giving (from the 2015 data) a value of 67.31 km s^{-1}/Mpc, I've always used a different value:

$$H_0 = 72\ \frac{\text{km s}^{-1}}{\text{Mpc}} \tag{23}$$

The Hubble parameter seems to be a deeply personal value, with people clinging desperately to the value that they've always used, so long as it's between the accepted bounds. Not that I'm any better; I insist on using 72 km s^{-1}/Mpc. The various (recent) observational values of H_0 are plotted in Figure 1, to give a sense of the variation in the measurements. For a few of the recent measurements, the error bars don't overlap, so the values cannot possibly be consistent.

Sometimes the Hubble parameter is defined in terms of a unitless quantity h, such that:

$$H_0 \equiv 100h\ \frac{\text{km s}^{-1}}{\text{Mpc}} \tag{24}$$

I'm not a huge fan of using h, so I will avoid it. But it's been around since the days when physicists were divided between $H_0 = 50$ km s^{-1}/Mpc and $H_0 = 100$ km s^{-1}/Mpc, so they defined the dimensionless quantity h and wrote out cosmological parameters in terms of h, since some important ones are dimensionless and can't be written in terms of H_0. Since we're *fairly* certain what the value of H_0 is now, there isn't as much a need for h.

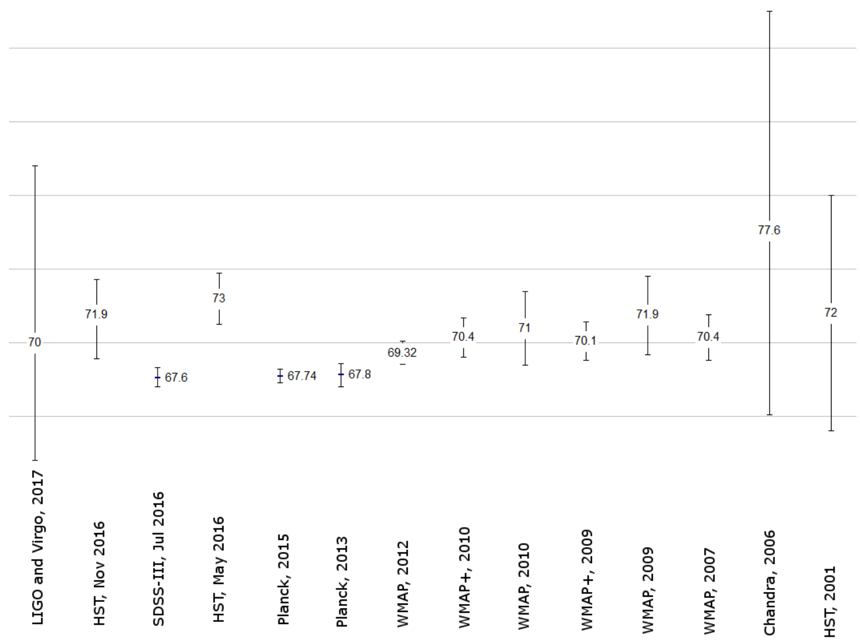

Figure 1: Recent measurements of H_0. *Image credit: WikimediaCommons, author Ewen.*

Now that we've defined the Hubble parameter, let's discuss the interpretation: it's very simple,

$$\begin{aligned} H(t) > 0 &\quad \Rightarrow \quad \text{the universe is expanding} \\ H(t) < 0 &\quad \Rightarrow \quad \text{the universe is collapsing} \\ H(t) \equiv 0 &\quad \Rightarrow \quad \text{the universe is static} \end{aligned} \tag{25}$$

If we take a look at the Friedmann equations, given in equation (21), we can define a new quantity, known as the acceleration parameter:

$$A(t) \equiv \frac{\ddot{a}}{a} \tag{26}$$

which has a very easy interpretation as well:

$$\begin{aligned} A(t) > 0 &\quad \Rightarrow \quad \text{the universe's expansion is accelerating} \\ A(t) < 0 &\quad \Rightarrow \quad \text{the universe's expansion is decelerating} \\ A(t) \equiv 0 &\quad \Rightarrow \quad \text{the universe's expansion is constant} \end{aligned} \tag{27}$$

It might be better to describe the implications of $A(t)$ in terms of the rate of change of the Hubble parameter, because if a universe is collapsing, $A(t) > 0$ means it will collapse more slowly and more slowly, etc., since it's not expanding. However, in the context of the cosmology of our particular universe, not in the context of a general Friedmann cosmology, it makes more sense to describe $A(t)$ as above, since our universe is observed to be expanding and we'll be interested to know how the rate of expansion acts over time.

We can re-write both Friedmann equations in terms of the two new parameters:

$$\begin{aligned} H^2(t) &= \frac{8\pi}{3}\rho - \frac{k}{a^2} \\ A(t) &= -\frac{4\pi}{3}(\rho + 3p) \end{aligned} \tag{28}$$

Notice $A(t)$ is determined entirely by the equation of state of a perfect fluid. Substituting $p = w\rho$, the sign of $A(t)$ is determined by:

$$3\left(\frac{1}{3} + w\right)\rho$$

If this value is greater than zero, then $A(t) < 0$ and the expansion is decelerating; if this value is less than zero, then $A(t) > 0$ and the expansion is acceleration; and if the value is zero, then the expansion is constant.

What does this parameter look like for our three different types of energy? Well, for matter, for which $w = 1$, we get:

$$3\left(\frac{1}{3} + w\right)\rho = 4\rho$$

which is strictly positive (since mass is strictly positive), and so if a universe filled only with matter were expanding, the expansion would be decelerating. This makes perfect sense: the gravity of the mass would pull the universe back in on itself, slowing down the expansion until eventually it collapsed back in on itself.

What about for radiation? Recall that $w = 1/3$ for radiation, so:

$$3\left(\frac{1}{3} + w\right)\rho = 2\rho$$

So, the same interpretation as for a matter universe; eventually the universe would collapse back in on itself.

What about for dark energy? Here's where we can really interpret dark energy, and what it's definition actually is. The above parameter, which determines the sign of $A(t)$, has a zero at:

$$w = -\frac{1}{3}$$

Any substance with $wle - 1/3$ is termed dark energy. I was being misleading before when I said dark energy had $w = -1$; this is technically only true if the dark energy is due to the cosmological constant Λ, which is why I've labeled everything to do with dark energy by the subscript Λ.

There are other (hypothetical) forms of dark energy. One idea is that of quintessence, a scalar field Q under the influence of a potential $V(Q)$ such that:

$$w = \frac{\frac{1}{2}\dot{Q}^2 - V(Q)}{\frac{1}{2}\dot{Q}^2 + V(Q)}$$

Quintessence is a dynamical form of dark energy, and $w = w(t)$ for quintessence. This is obviously different from the cosmological constant, for which $w = -1$ independent of time. Quintessence is closely related to the scalar field that is thought to drive cosmological inflation, something called a "slow-rolling field." While I won't get into inflation, it's simple to show in the theory of inflation that a slow-rolling field will definitely cause an accelerated expansion, thus fulfilling the definition of dark energy as something with $w \leq -1/3$ (i.e. something that produces $A(t) > 0$).

Einstein formulated the cosmological constant such that his field equation read:

$$R_{\mu\nu} - \left(\frac{1}{2}R - \Lambda\right) g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{29}$$

Another way to interpret Λ is to move it over to the right-hand-side and set $T_{\mu\nu} = 0$, i.e. consider a vacuum. Then, the field equation above becomes:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \Lambda g_{\mu\nu}$$

Clearly, Λ acts as a sort of vacuum energy; something that can affect the geometry of the universe without any matter present. While the most accepted theory of cosmology, the ΛCDM theory[3], there are serious issues with the cosmological constant as the source of dark energy. The main issue has to due with out inability to estimate Λ from a field theoretic perspective. It's a commonly cited result that in field theory, the estimated value of Λ is off by 120 orders of magnitude.

So, now that we understand dark energy a bit better (though we haven't explored a physical interpretation of what $w < 0$ means), we can now show why $w = -1$ for the cosmological constant. Note that, from now on, I'm going back to calling it dark energy; it's the most widely accepted idea for the source of dark energy, and it's just simpler to call it dark energy than "the cosmological constant" every time I meantion it. Just be aware of the caveat.

The modified Einstein field equation, with Λ, given by equation (29), modifies the first Friedmann equation from equation (21) such that:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi}{3}\rho$$

We can define the dark energy density, ρ_Λ, such that:

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi} \tag{30}$$

Notice that, since Λ is a constant (by definition; it's why it's called the cosmological *constant*), the energy density ρ_Λ must, too, be a constant. We already saw that $w = -1$ defines the equation of state for a fluid that has a constant energy density, so our previously unjustified choice is now justified by the definition of the cosmological constant.

What else can we explore about cosmology? Something that was skipped over in the definition of the Hubble parameter were its funky units. They are typically given as $[H(t)] = \text{km s}^{-1}/\text{Mpc}$. There's a practical reason for this: Hubble's law was historically given, and still interpreted, as a relationship between speed and distance:

$$v = H(t)d \tag{31}$$

where $d = a$ and $v = \dot{a}$. Cosmological distances are often measured in Mpc (or hundreds of Mpc, which is typically the distance scale that defines "cosmological distance), and the speeds of galaxies (moving away from us, since the universe is expanding) are typically on the order of km/s, so it just makes sense, in the context of the above equation, to use the seemingly-funky units of km s^{-1}/Mpc.

However, notice that since $[v] = [Hd]$, that the units of the Hubble parameter are inverse time. So H_0^{-1} should represent some time measurement. But what measurement exactly? Figure 2 plots $a(t)$ vs. time, and uses the plot to define H_0^{-1}.

[3]Standing for Λ-cold dark matter. Dark matter is nothing like dark energy; it's simply a source of mass that we haven't been able to observe. This is most likely due to the fact that it doesn't interact with photons, which makes it an exotic form of matter that we haven't seen before, since neutrinos are too light to be responsible for dark matter.

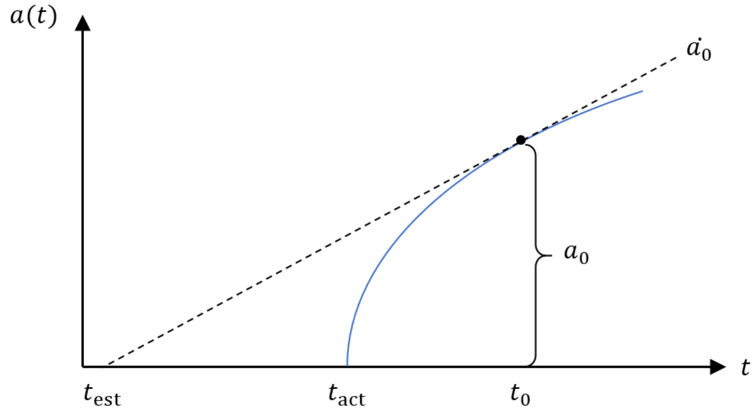

Figure 2: Plot of $a(t)$ vs. t, allowing for a natural interpretation of H_0^{-1}.

The dashed line in the figure represents a constant slope, i.e. if we assume that H_0 was constant throughout time. With $\dot{a}_0$ clearly defining the slope of this line, we can say:

$$a(t) = \dot{a}_0 t_{est}$$

where t_{est} is the estimated time since the big bang. As a side note, times are often defined in the opposite direction one is used to, because cosmology is all about look *backwards* in time. For instance, in Figure 2, t_0 would be set to zero by convention, and so the point at which the dashed line intersected the time-axis would be $-t_{est}$. This still means there has been a time $+t_{est}$ since $a(t) = 0$, which is considered the beginning of the universe[4]. Simply solving the above equation for t_{est} gives us $a(t=0)/\dot{a}_0$, or the current scale factor divided by the current expansion rate, which gives us:

$$t_{est} = H_0^{-1} \tag{32}$$

That is, the interpretation of H_0^{-1} is precisely the age of the universe *if* the acceleration of expansion was zero for the universe; i.e. the Hubble parameter is always $H(t=0) = H_0$. In reality this isn't true, though. The modern picture of the evolution of the universe has an initial period of rapid inflation, followed by a deceleration of that inflation, as shown in Figure 2. This means that:

$$t_{est} > t_{act} \tag{33}$$

i.e. we're over-estimating the age of the universe by simply inverting H_0.

How does the estimate compare to more precise computations of the age of the universe? Since we aren't ready to perform the actual computation of the age of the universe at this point, I'll simply present the value that Planck has measured. This means that we have to use $H_0 = 67$ km s^{-1}/Mpc, though, since that's the value Planck has measured. Converting to SI units, $H_0 = 2.171 \times 10^{-18}$ s^{-1}, which means that $H_0^{-1} = 4.61 \times 10^{17}$ s, or:

$$t_{est} = 14.61 \text{ Gyr} \tag{34}$$

The precise estimates of the age of the universe, however, put it at:

$$t_{act} = 13.82 \text{ Gyr} \tag{35}$$

So, as expected (though I didn't justify the shape of $a(t)$ in Figure 2; that'll come later), $t_{est} > t_{act}$.

[4]Though we won't get into it, $a(t) = 0$ a point known as a singularity; this is the starting point of the universe in the big bang model of evolution.

4 Scale-Factor Evolution in Friedmann Cosmologies

Let's take a look now at how geometry and energy density affects the evolution of the scale-factor in Friedmann cosmologies. We'll be performing this entire analysis from the perspective of a universe that has $H_0 > 0$, like our universe. Let's recall the Friedmann equations written in terms of the two parameters, $H(t)$ and $A(t)$, given by equation (28):

$$H^2(t) = \frac{8\pi}{3}\rho - \frac{k}{a^2}$$

$$A(t) = -\frac{4\pi}{3}(\rho + 3p)$$

Since $H(t)$ is already positive, at this moment, the only way for the universe to start collapsing is if $H(t)$ goes to zero first. Without that, there's no way for $H(t)$ to change signs (since it must be a continuous function). Recall the general equation for energy density, given by equation (15):

$$\rho = Ca^{-3(1+w)}$$

As long as $1 + w > 0$, then $\rho \to 0$ as $a \to \infty$; i.e. ρ decreases monotonically with a. This means that, since $H_0 > 0$, as $a \to \infty$

$$H^2(t) \to -\frac{k}{a^2}$$

If $k = -1$ or $k = 0$, then $H(t)$ will never reach 0 in a finite amount of time, and so $H(t)$ could never flip signs, and so the universe could never collapse back on itself. That is,

> **Flat or hyperbolic universes, under Friedmann cosmologies, will never collapse back on themselves if they have $H_0 > 0$; the only fate for them is to keep expanding perpetually, with $H(t) \to 0$ as $t \to \infty$.**

What about if $k = +1$, though? Well, clearly $H^2(t)$ can never be negative, so at some point, as ρ is decreasing, there must have been a time t' when $H(t') = 0$. This means that the expansion is decelerating, so $A(t) < 0$. This is certainly possible, for:

$$\rho + 3p = \rho(1 + 3w) > 0 \quad \Rightarrow \quad w > -\frac{1}{3}$$

We already said for ρ to go to decrease as a increased, $w > -1$, so this restriction on w doesn't conflict with our previous restriction, and everything is good so far. So, assuming $w > -1/3$, eventually there will be a time t' when $H(t') = 0$, and since $A(t) < 0$ for all time, $H(t)$ will flip signs and become negative, causing the universe to collapse in on itself. Since $A(t)$ is strictly negative, the rate at which the universe collapses will accelerate, collapsing the universe back down to its starting point: $a(t) = 0$. So:

> **The only geometry for a universe under Friedmann cosmology to collapse back in on itself is that of a sphere. As long as $w > -1$, the universe has a possibility of doing so. If $w > -1/3$, the universe will definitely collapse back in on itself, and fall back to $a(t) = 0$.**

But notice something: all of this analysis was for a *single* fluid, with a fixed w. What about a physical universe, like ours, which has 3 different types of fluids, each with its own value of w? Well, the analysis gets more complicated, obviously, but there's still a lot to learn from a basic overview of the Friedmann equations.

The first thing we need to do is recognize that geometry plays a key role in the evolution of the universe. Keeping this in mind, we'll define a critical density ρ_c such that $k = 0$ in the Friedmann equation for H^2:

$$\rho_c \equiv \frac{3H^2}{8\pi} \tag{36}$$

Note that since H^2 is a function of time, so is ρ_c. The critical density gives us a really convenient way to determine the geometry of any universe under Friedmann cosmology. At our current time $t = 0$, we have a Hubble parameter of H_0 and thus a critical density of $\rho_{c,0} = 3H_0^2/8\pi$. The actual density at the current time, ρ_0 then uniquely determines the geometry of the universe. We can manipulate the Friedmann equation for H^2 to see this (note that this will be defined at our current time):

$$H_0^2 = \frac{8\pi}{3}\rho_0 - \frac{k}{a_0^2} \quad \Rightarrow \quad \frac{k}{a_0^2} = \frac{8\pi}{3}\left(\rho_0 - \frac{3H_0^2}{8\pi}\right)$$

Thus, we have:

$$\frac{k}{a_0^2} = \frac{8\pi}{3}(\rho_0 - \rho_{c,0}) \tag{37}$$

If the current energy density of the universe, ρ_0, is greater than the current critical density, $\rho_{c,0}$, then the universe is spherical ($k > 0$); if $\rho_0 < \rho_{c,0}$, then the universe is hyperbolic ($k < 0$); and if $\rho_0 = \rho_{c,0}$, then the universe is flat ($k = 0$).

So, clearly the energy density ρ (at any time) is critically important to understanding the evolution of the universe, since it determines the geometry, and the geometry determines how the expansion will accelerate or decelerate. Since we don't have one type of fluid in our universe, but three types, it's natural to simple define the total energy density as:

$$\rho \equiv \rho_m + \rho_r + \rho_\Lambda \tag{38}$$

where, once again, the subscript m is for matter, r is for radiation, and Λ is for dark energy.

Notice something else, though: the actual energy densities are irrelevant. Look back at equation (37), and divide by the critical density $\rho_{c,0}$. This yields:

$$\frac{k}{\rho_{c,0}a_0^2} = \frac{8\pi}{3}\left(\frac{\rho_0}{\rho_{c,0}} - 1\right)$$

If we define a *relative* energy density:

$$\Omega \equiv \frac{\rho}{\rho_c} \tag{39}$$

with analogous definitions for the individual fluids, and analogous definitions at the current time $t = 0$, then the above equation becomes:

$$\frac{k}{\rho_{c,0}a_0^2} = \frac{8\pi}{3}(\Omega_0 - 1) = \frac{8\pi}{3}(\Omega_{m,0} + \Omega_{r,0} + \Omega_{\Lambda,0} - 1) \tag{40}$$

Since we know that $\rho_{c,0}$ is positive[5], it's not the (current) energy density ρ_0 that determines k, but the (current) *relative* energy density Ω_0 that does. If we could measure Ω_0, we would know exactly what the geometry of the universe was, i.e. what k was, and then we could predict how $a(t)$ would change based on how Ω changes with time.

[5]Look back at equation (36); since H^2 is always positive, ρ_c will always be positive.

Before continuing, I am oversimplifying the analysis a bit, but this is how you learn cosmology: you start with a super simplification (e.g. only one type of fluid), then a lesser simplification (e.g. what we're doing now), and then work your way up to the real thing. The main part of the analysis that I'm completely ignoring is the fact that k is *not* a constant in time; the geometry of the universe can absolutely change over time. This is as easy to see as looking as the equation we've been using: since Ω is a function of time, then there is no reason to assume that $\Omega(t)-1$ will *never* change sign[6]. It turns out that this isn't very relevant to our universe, especially not at the level we're considering cosmology at, so we can continue despite this omission.

Getting back to it, it turns out that we can, actually, measure the energy content of the universe. The Planck satellite has been able to measure it extremely precisely, to a value of (as of the 2013 results):

$$\Omega_0 = 1.0005 \pm 0.0033 \tag{41}$$

This is insanely close to 1; for all intents-and-purposes, the universe is flat. We can define the relative density for any fluid i as:

$$\Omega_i \equiv \frac{8\pi}{3H_0^2}\rho_i = \frac{8\pi}{3H_0^2}\rho_{i,0}a^{-3(1+w_i)} \tag{42}$$

where I used equation (15) for $\rho_{i,0}$ in the last equality. Though I didn't mention it before, $C_{i,0}$ has the interpretation of the initial density of fluid i. This is because a_0 is defined to be 1, so $\rho_{i,0} = C_{i,0}a_0^{-3(1+w_i)} = C_{i,0}$, giving it the meaning I just used for it. Thus, the relative density, as function of scale-factor, can be expressed, with the appropriate values of w_i, as:

$$\Omega = \frac{8\pi}{3H_0^2}\left(\rho_{i,0}a^{-3} + \rho_{r,0}a^{-4} + \rho_{\Lambda,0}\right)$$

Or, better yet, re-writing this in terms of the current relative densities $\Omega_{i,0}$:

$$\Omega = \Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4} + \Omega_{\Lambda,0} \tag{43}$$

Now we're prepared to discuss what happened to the different densities of the different fluids as the universe was expanding from $a(t)=0$. We can see that the fastest energy source to die off is radiation, since it goes like a^{-4}. As Ω_r is dropping, both Ω_m and Ω_Λ are rising, because they are *relative* quantities; Ω_m rises slightly due simply to the drop in Ω_r, despite ρ_m continuously dropping as a^{-3}, but Ω_Λ rises due to the fact that ρ_Λ is a constant. Fairly quickly, $\Omega_r \to 0$ for all intents-and-purposes, so we ignore whatever effects radiation might have on the evolution of the universe at present times. As time continues to pass, as long as a continues to grow (which it will unless the universe is spherical; recall that our universe is flat), then the matter density is eventually going to die off too, since ρ_m continuously decreases while ρ_Λ remains a constant. Thus, at some point, $\Omega_m \to 0$, and the only energy source left will be dark energy, i.e. $\Omega = \Omega_\Lambda$.

Take a look at Figure 3, which has the actual initial conditions of our universe[7]. In the initial universe (a short time after because of cosmic inflation, which we won't worry about), Ω_m and Ω_r were basically the same value, with ρ_r only slightly larger. However, since ρ_r drops off faster

[6]Note that only with a change in sign of k will the geometry change; this was the whole thing about the Robertson-Walker being reparametrizable with a change in the scale of k, so that the geometry was only sensitive to the sign.

[7]In cosmology, we don't compute things from the initial conditions at the beginning of the universe, and then move forward in time; we start at our current time $t=0$, and then work backwards through time to derive the conditions at the start of the universe. But for our purposes, it's better to think that someone was around at the start of the universe to measure the relative densities Ω_i, so that we could then predict how they would change from the start of the universe to the present time, when they have a value of $\Omega_{i,0}$.

than ρ_m, Ω_m caught up to Ω_r fairly quickly, and the universe transitioned from being radiation-dominated (meaning the major contributor to Ω, or the effective equation of state w_{eff}, if you'd like) to being matter-dominated. However, the matter density continued to drop, but the dark energy density never did; it remained a constant. At some time, then, there was (or will be) a transition from a matter-dominated universe to a dark energy-dominated universe. Current estimates put this transition not too long ago, as you can see in Figure 3 (current time, measured since the big bang, is on the order of 10^{10} years).

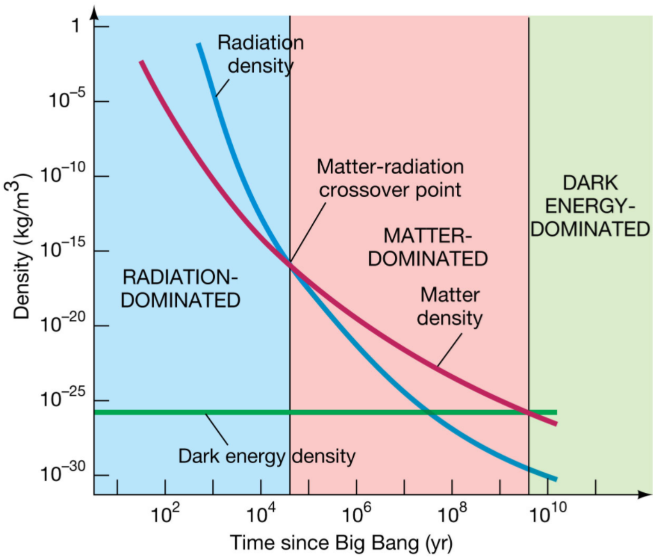

Figure 3: Time-evolution of energy densities. *Image credit: Prof. Jim Brau, University of Oregon.*

Why does it matter what energy source dominates the universe? Well, not worrying about k changing sign on us, the type of matter that dominates the universe controls what the effective value of w is going to be in the equation of state, some w_{eff} let's say. This has huge implications on how the scale factor is going to evolve, as we've discussed before. In our simple analysis, we approached the evolution of the universe for a single type of energy source; well, that's exactly the same as we're doing here. The early universe was dominated by radiation, so $w_{eff} \approx 1/3$, and $a(t)$ would evolve like for a universe with only radiation. The majority of the time since the big bang has been matter-dominated, though, so w_{eff} has been approximately 0 throughout this time. Now that we're transitioning, or have transitioned, into a dark energy-dominated universe, from now on $w_{eff} \approx -1$.

For both the radiation-dominated and matter-dominated universes, we see from the Friedmann equation for $A(t)$, given in equation (28), that:

$$A(t) = -\frac{4\pi}{3}(\rho + 3p) = -\frac{4\pi}{3}\rho(1 + 3w) < 0$$

since $w = 1/3$, so $1 + 3w = 2$ for a radiation-dominated universe, and $w = 1$, so $1 + 3w = 4$ for a matter-dominated universe. This means that for the majority of the time since the beginning of the universe, the rate at which the universe has been expanding has been decreasing. This is exactly what was depicted in Figure 2, but now we have a reason to expect this result.

But, once the universe transitions to dark energy-dominated, all hope of not expanding out to infinity is lost. Not only will a dark energy-dominated universe continue to expand forever, but it will continue to accelerate its expansion since $w = -1$ for dark energy, so $1 + 3w = -2$, and thus

$A(t)$ is strictly positive. The moment our universe became dark energy-dominated, our fate has been essentially sealed.

At what point should we expect the transition from matter-dominated to dark energy-dominated? This can be answered by analyzing, once again, our Friedmann equation for $A(t)$. Expanding both ρ and p in terms of matter and dark energy terms (we can ignore radiation, since ρ_r long ago dropped to practically zero), such that:

$$A(t) = -\frac{4\pi}{3}(\rho_m + \rho_\Lambda + 3p_m + 3p_\Lambda)$$

Since $p_m = 0$ and $p_\Lambda = -\rho_\Lambda$, the above equation reduces to:

$$A(t) = -\frac{4\pi}{3}(\rho_m - 2\rho_\Lambda) \tag{44}$$

near the transition between matter-dominated and dark energy-dominated. We know that, along with this transition, comes the change from a decelerating expansion to an accelerating expansion. So the change itself should occur when $A(t)$ is transition from negative to positive, i.e. when it's zero. Thus, the transition from a matter-dominated to a dark energy-dominated universe occurs when:

$$\frac{\rho_\Lambda}{\rho_m} = \frac{1}{2} \tag{45}$$

The density of dark energy simply has to reach 1/2 the value of the matter density for the universe to become dark energy-dominated; they don't even have to be equal! With respect to this result, Figure 3 is slightly off, but it's still fine to illustrate the major points being made.

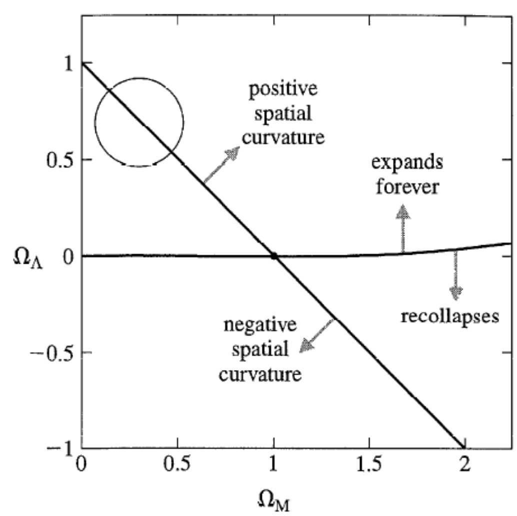

Figure 4: What the dynamics of our universe should be, based on observations of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$. *Image credit: Carroll (2004).*

To wrap it up, I'll simply present a figure that summarizes a lot of ideas discussed, but with the added dimension that the geometry, i.e. k, can also change over time. Figure 4[8] above plots $\Omega_{\Lambda,0}$ vs. $\Omega_{m,0}$, and shows how the dynamics of our universe will change based on how the parameters change. The diagonal line represents a constant line $\Omega_0 = 1$, which guarantees a flat universe (i.e.

[8]Taken from Carroll (2004).

$k = 0$); any diagonal line underneath will be a line of constant $\Omega_0 < 1$, so the universe will be hyperbolic (i.e. $k < 0$); and any diagonal line above will be a line of constant $\Omega_0 > 1$, so the universe would be spherical (i.e. $k > 0$). Note that whenever I say a "diagonal line," I mean one with a 45^o slope like the one drawn in the Figure.

Current Planck observations (from the 2015 data release) put the relative densities at:

$$\begin{aligned}\Omega_{\Lambda,0} &= 0.685 \pm 0.013 \\ \Omega_{m,0} &= 0.315 \pm 0.013\end{aligned} \tag{46}$$

These sum to exactly 1, with an error of ± 0.013. The trouble is that, along with almost everything in cosmology, there are a few different ways to measure these densities, and the physicists community isn't convinced about which is the most accurate. This is why Figure 4 has a bubble around $\Omega_{\Lambda,0} = 0.7$ and $\Omega_{m,0} = 0.3$; there are differing observations[9] that provide differing values, but they all have logical validity, and no one has been able to rule out a particular method for determining the values. Until that time comes, we'll be perpetually living in this bubble, unsure about the exact geometry of the universe or its exact fate.

A Equations of State for Radiation and Matter

We know from classical kinetic theory that the pressure on a surface p_s due to a beam of particles at a speed v with a density n arriving at an angle of θ relative to the normal of the surface is

$$p_s = n\gamma m v^2 \cos^2\theta \tag{47}$$

where I have added a factor of γ to make the equation relativistically correct. If want to find the pressure on the inner surface of a sphere due to an isotropic distribution of particle velocities, we simply need to average $\cos^2\theta$ over the unit sphere:

$$\langle \cos^2\theta \rangle = \frac{\int \cos^2\theta d\Omega}{\int d\Omega} = \frac{2\pi \int_{-1}^{1} \cos^2\theta d(\cos\theta)}{4\pi} = \frac{1}{2}\left(\frac{1}{3}x^3\right)\Big|_{-1}^{1} = \frac{1}{3}$$

So, the pressure exerted by the gas of particles with an isotropic velocity distribution is

$$p = \frac{1}{3} n\gamma m v^2 \tag{48}$$

Now, we know that the relativistic energy of a single particle is going to be

$$E = \sqrt{p^2 + m^2} = \sqrt{\gamma^2 m^2 v^2 + m^2} = m\sqrt{\gamma^2 v^2 + 1}$$

Note that $\gamma^2 v^2 + 1 = \gamma^2$. If we multiply the above equation by the particle density n, we get the energy density:

$$\rho = nm\sqrt{\gamma^2} = n\gamma m \tag{49}$$

So, our equation for the pressure becomes

$$p = \frac{1}{3}\rho v^2 \tag{50}$$

[9]The three major ones that contribute to this are CMB observations, like those of Planck, supernovae type Ia observations, and galactic cluster observations. Hopefully one day we'll be able to rule on which of the three is the best method for measuring the densities, or at least maybe get all three to agree to greater precision.

For ultra-relativistic particles, like photons, $v \to 1$ and

$$p_r = \frac{1}{3}\rho \tag{51}$$

where the r is for radiation, which is what photons are typically referred to in cosmology. For non-relativistic particles, like a cold gas, $v \to 0$ and

$$p_m = 0 \tag{52}$$