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Maximally Symmetric Spaces

Douglas H. Laurence

Department of Physical Sciences, Broward College, Davie, FL 33314

Abstract: These notes follow Weinberg's derivation of the Riemann Curvature tensor for a maximally symmetric space presented in "Gravitation and Cosmology," with some details filled in. Weinberg is particularly difficult to read, so writing down these notes and filling in the gap really helped me understand his derivation, which is the most rigorous and thorough derivation in any of the popular general relativity or cosmology textbooks. These notes start by presenting Killing vectors from scratch, and then apply the most restrictions on can on the form of the Riemann curvature tensor for a maximally symmetric space. Specifically, the maximally symmetric space considered is a homogeneous and isotropic space, which is obviously important when studying cosmology.

Contents

1 Isometries of the Metric

A metric $g_{\mu\nu}(x)$ is **form-invariant** under some coordinate transformation $x \mapsto x'$ when the transformed metric $g'_{\mu\nu}(x')$ is the same function of its argument x' as the original metric $g_{\mu\nu}$ was of its argument x. The metric transformation is defined by:

$$g_{\mu\nu}(x) = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g'_{\rho\sigma}(x') \tag{1.1}$$

The above definition of a form-invariant metric is equivalent to the statement:

$$g'_{\rho\sigma}(x') = g_{\rho\sigma}(x') \tag{1.2}$$

i.e. that the form of the metric is invariant under the transformation; the function itself remains the same. So, equation (1.1) becomes:

$$g_{\mu\nu}(x) = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x') \tag{1.3}$$

What we want to consider are **isometries**, which are the coordinate transformations $x \mapsto x'$ that leave a metric form-invariant. We'll restrict ourselves to infinitesimal isometries, as all the interesting physics is going to be related to infinitesimal transformations. An infinitesimal coordinate transformation can be written as:

$$x^{\mu} \mapsto x'^{\mu} = x^{\mu} + \epsilon \xi^{\mu} \qquad (\epsilon \ll 1) \tag{1.4}$$

We can plug this transformation into equation (1.3) and keep terms up to order ϵ. First, the partial derivatives will be:

$$\frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \left(\frac{\partial x^{\rho}}{\partial x^{\mu}} + \epsilon \frac{\partial \xi^{\rho}}{\partial x^{\mu}} \right) \left(\frac{\partial x^{\sigma}}{\partial x^{\nu}} + \epsilon \frac{\partial \xi^{\sigma}}{\partial x^{\nu}} \right) = \delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} + \epsilon \delta^{\rho}_{\mu} \frac{\partial \xi^{\sigma}}{\partial x^{\nu}} + \epsilon \delta^{\sigma}_{\nu} \frac{\partial \xi^{\rho}}{\partial x^{\mu}} + \mathcal{O}(\epsilon^2)$$

Next, taking the Taylor expansion of $g_{\rho\sigma}(x')$ to order ϵ:

$$g_{\rho\sigma}(x') = g_{\rho\sigma}(x) + \epsilon\xi^\alpha \frac{\partial g_{\rho\sigma}(x)}{\partial x^\alpha} + \mathcal{O}(\epsilon^2)$$

Plugging these two results into equation (1.3), we see that:

$$\begin{aligned}
g_{\mu\nu}(x) &= \left(\delta^\rho_\mu \delta^\sigma_\nu + \epsilon\delta^\rho_\mu \frac{\partial \xi^\sigma}{\partial x^\nu} + \epsilon\delta^\sigma_\nu \frac{\partial \xi^\rho}{\partial x^\mu}\right)\left(g_{\rho\sigma}(x) + \epsilon\xi^\alpha \frac{\partial g_{\rho\sigma}(x)}{\partial x^\alpha}\right) \\
&= \delta^\rho_\mu \delta^\sigma_\nu g_{\rho\sigma}(x) + \epsilon\delta^\rho_\mu \frac{\partial \xi^\sigma}{\partial x^\nu} g_{\rho\sigma}(x) + \epsilon\delta^\sigma_\nu \frac{\partial \xi^\rho}{\partial x^\mu} g_{\rho\sigma}(x) + \delta^\rho_\mu \delta^\sigma_\nu \epsilon\xi^\alpha \frac{\partial g_{\rho\sigma}(x)}{\partial x^\alpha} \\
&= g_{\mu\nu}(x) + \epsilon\frac{\partial \xi^\sigma}{\partial x^\nu} g_{\mu\sigma}(x) + \epsilon\frac{\partial \xi^\rho}{\partial x^\mu} g_{\rho\nu}(x) + \epsilon\xi^\alpha \frac{\partial g_{\mu\nu}(x)}{\partial x^\alpha}
\end{aligned}$$

Thus, to order ϵ, the condition for form-invariance of the metric for an infinitesimal isometry is equivalent to the following condition on the metric:

$$\frac{\partial \xi^\sigma}{\partial x^\nu} g_{\mu\sigma}(x) + \frac{\partial \xi^\rho}{\partial x^\mu} g_{\rho\nu}(x) + \xi^\alpha \frac{\partial g_{\mu\nu}(x)}{\partial x^\alpha} = 0 \tag{1.5}$$

From now on, I'm going to drop the explicit dependence on x, since all of the metrics are being evaluated at the same position, so there is no longer a need to be explicit.

The next step is to notice that:

$$\frac{\partial}{\partial x^\nu}(\xi^\sigma g_{\mu\sigma}) = \frac{\partial \xi^\sigma}{\partial x^\nu} g_{\mu\sigma} + \xi^\sigma \frac{\partial g_{\mu\sigma}}{\partial x^\nu}$$

So equation (1.5) becomes:

$$\frac{\partial}{\partial x^\nu}(\xi^\sigma g_{\mu\sigma}) - \xi^\sigma \frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial}{\partial x^\mu}(\xi^\rho g_{\rho\nu}) - \xi^\rho \frac{\partial g_{\rho\nu}}{\partial x^\mu} + \xi^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} = 0 \tag{1.6}$$

There are a couple of things to note about the above equation. There are two non-dummy indices: μ and ν. Besides those, all indices are dummy indices. In the second term, I want to replace σ with α, and in the fourth term, ρ with α. Another thing to notice is that:

$$\xi^\sigma g_{\mu\sigma} = \xi_\mu$$

So, equation (1.6) becomes:

$$\frac{\partial \xi_\mu}{\partial x^\nu} + \frac{\partial \xi_\nu}{\partial x^\mu} + \xi^\alpha \left(\frac{\partial g_{\mu\nu}}{\partial x^\alpha} - \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\alpha\nu}}{\partial x^\mu}\right) = 0 \tag{1.7}$$

Recalling the definition of the Christoffel symbols of the first kind:

$$\Gamma_{\alpha\mu\nu} = \frac{1}{2}\left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)$$

equation (1.7) becomes:

$$\frac{\partial \xi_\mu}{\partial x^\nu} + \frac{\partial \xi_\nu}{\partial x^\mu} - 2\xi^\alpha \Gamma_{\alpha\mu\nu} = 0 \tag{1.8}$$

Noting that $\xi^\mu = x_\lambda g^{\mu\lambda}$ and that the Christoffel symbol of the second kind is defined as:

$$\Gamma^\lambda_{\mu\nu} = g^{\lambda\alpha}\Gamma_{\alpha\mu\nu}$$

equation (1.8) becomes:

$$\frac{\partial \xi_\mu}{\partial x^\nu} + \frac{\partial \xi_\nu}{\partial x^\mu} - 2\xi_\lambda \Gamma^\lambda_{\mu\nu} = 0 \tag{1.9}$$

The factor of 2 in front of the Christoffel term allows that term to be split, one for each derivative of ξ, which turns those ordinary derivatives into covariant derivatives. Thus, we arrive at the equation:

$$\nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu = 0 \tag{1.10}$$

2 Killing Vectors

The equation derived at the end of the last section is known as the **Killing equation**, and the vector fields ξ that satisfy it are known as **Killing vector fields**; they define the particular isometries of the metric that one might be interested in. For instance, we could construct a killing vector to describe a translational isometry or a rotational isometry.

What Killing vectors allow us to do is reduce the problem of finding all symmetries of a particular metric down to simply finding the corresponding Killing vectors. Sometimes this problem is very easy, for instance if a metric has an explicit independence of a coordinate. For example, the Minkowski metric is invariant under a translation of any of its four coordinates, so there would be one Killing vector per coordinate (at least).

Note that a linear combination of Killing vectors is also a Killing vector, since it will solve the Killing equation like any linear combination of solutions to any differential equation. So, technically, it's the spanning set of Killing vectors is actually what describes the isometries of a particular metric.

The Killing equation is actually more useful than it appear at first sight; it will allow us to define a Killing vector at any point x if we only know the Killing vector at a specific point X and its covariant derivative at that point X. It turns out that we can show that second-order covariant derivatives of Killing vectors are not unique, but are proportional to the Killing vector itself, so all second- and higher-order covariant derivatives of the Killing vector can be expressed in terms of just the Killing vector itself and its first-order covariant derivative. Thus a Taylor expansion for a Killing vector at x can be expressed entirely in terms of the Killing vector at X and its first-order covariant derivative at X.

To show this, we need to begin with an identity for second-order covariant derivatives (a la Weinberg, "Gravitation and Cosmology," (13.1.6) or Dirac, "The General Theory of

Relativity," (11.2)). From now on, I will use the notation that a covariant derivative is given by a subscript with a semi-colon preceding it. The relevant identity is:

$$\xi_{\mu;\nu;\rho} - \xi_{\mu;\rho;\nu} = -R^{\lambda}_{\mu\nu\rho}\xi_{\lambda} \tag{2.1}$$

Combining this with the first Bianchi identity of the Riemann curvature tensor:

$$\begin{aligned} & R^{\lambda}_{\mu\nu\rho} + R^{\lambda}_{\nu\rho\mu} + R^{\lambda}_{\rho\mu\nu} = 0 \\ \Rightarrow\ & R^{\lambda}_{\mu\nu\rho}\xi_{\lambda} + R^{\lambda}_{\nu\rho\mu}\xi_{\lambda} + R^{\lambda}_{\rho\mu\nu}\xi_{\lambda} = 0 \\ \Rightarrow\ & \xi_{\mu;\nu;\rho} - \xi_{\mu;\rho;\nu} + \xi_{\nu;\rho;\mu} - \xi_{\nu;\mu;\rho} + \xi_{\rho;\mu;\nu} - \xi_{\rho;\nu;\mu} = 0 \end{aligned}$$

Now we want to group the above Killing vector derivatives by the second-derivative, i.e. $\xi_{\mu;\nu;\rho}$ and $-\xi_{\nu;\mu;\rho}$ are to be grouped:

$$(\xi_{\mu;\nu;\rho} - \xi_{\nu;\mu;\rho}) + (\xi_{\nu;\rho;\mu} - \xi_{\rho;\nu;\mu}) + (\xi_{\rho;\mu;\nu} - \xi_{\mu;\rho;\nu}) = 0 \tag{2.2}$$

If we take the Killing equation and take a second covariant derivative:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \quad \Rightarrow \quad \xi_{\mu;\nu;\rho} + \xi_{\nu;\mu;\rho} = 0$$

we can substitute these modified Killing equations into equaton (2.2), yielding:

$$2\xi_{\mu;\nu;\rho} - 2\xi_{\rho;\nu;\mu} - 2\xi_{\mu;\rho;\nu} = 0$$

or:

$$\xi_{\mu;\nu;\rho} - \xi_{\mu;\rho;\nu} = \xi_{\rho;\nu;\mu} \tag{2.3}$$

Notice above that I chose a very particular set of signs (first term positive, second and third terms negative). This is because, as you can see in the equation immediately above, this gives the term $\xi_{\mu;\nu;\rho} - \xi_{\mu;\rho;\nu}$, which we have our general second-order covariant derivative identity for. The sign of the third Killing vector derivative isn't important, but the signs of the first two are.

Plugging this result into equation (2.1), we arrive at:

$$\xi_{\rho;\nu;\mu} = -R^{\lambda}_{\mu\nu\rho}\xi_{\lambda} \tag{2.4}$$

Thus, the second-order covariant derivative of a Killing vector depends on the Killing vector itself! This is a direct consequence of the Killing equation, because aside from that, we used an identity that is satisfied by any vector. Using the above equation, we can construct higher-order covariant derivatives in terms of the Killing vector itself and its first-order covariant derivative, allowing us to construct the Taylor expansion for a Killing vector at any point x knowing only the Killing vector at some point X and its first-order covariant derivative at that same point X.

So, any Killing vector can be written as the Taylor expansion:

$$\xi_\mu(x) = A_\mu^\lambda(x; X)\xi_\lambda(X) + B_\mu^{\lambda\nu}(x; X)\xi_{\lambda;\nu}(X) \tag{2.5}$$

where the functions A and B contain all the higher-order terms in the Taylor expansion that, through the relationship between the second-order derivative and the Killing vector itself, reduce to either the Killing vector or the first-order covariant derivative of the Killing vector. The functions A and B should, in general, depend on the metric and on the choice of point X, but should not depend on the initial value of the Killing vector $\xi_\lambda(X)$ or its derivative $\xi_{\lambda;\nu}(X)$, so are the same functions for *any* Killing vector.

We could broaden the above equation to allow for a set of Killing vectors $\{\xi^{(n)}\}$, in which case equation (2.5) be:

$$\xi_\mu^{(n)}(x) = A_\mu^\lambda(x; X)\xi_\lambda^{(n)}(X) + B_\mu^{\lambda\nu}(x; X)\xi_{\lambda;\nu}^{(n)}(X) \tag{2.6}$$

While the initial values of the Killing vector an its derivative will change with each individual Killing vector considered, the functions A and B will not, because they are independent of these initial values. Note that to avoid confusion, I indexed each individual Killing vector with (n) instead of a bare n, to avoid assuming that n is an index and the assumption that ξ_μ^n is a tensor quantity.

Now that we can describe all Killing vectors $\xi_\mu^{(n)}(x)$ based on their individual initial conditions, we need to consider only the linearly-independent Killing vectors so we can form the spanning set of Killing vectors which, as mentioned, actually describes the isometries of the metric. As always, a set of vectors $\{\xi_\mu^{(n)}\}$ is linearly-inependent if

$$\sum_n c_n \xi_\mu^{(n)}(x) = 0$$

if and only if the constant coefficients c_n are each independently 0.

3 Maximally Symmetric Spaces

The question is now: "What is the maximum number of linearly-independent Killing vectors one can have in an N-dimensional space?" Looking back at the Taylor expansion for $\xi_\mu^{(n)}(x)$, any Killing vector is defined by its initial conditions $\xi_\lambda^{(n)}(X)$ and $\xi_{\lambda;\nu}^{(n)}(X)$. In an N-dimensional space, the Killing vector $\xi_\lambda^{(n)}(X)$ has N independent components. Because of the Killing equation, the first-order covariant derivative of a Killing vector $\xi_{\lambda;\nu}^{(n)}(X)$ is anti-symmetric about its indices λ and ν, meaning that in N-dimensions it has $N(N-1)/2$ independent components. So a Killing vector $\xi_\mu^{(n)}(x)$ is defined by

$$N + \frac{N(N-1)}{2} = \frac{N(N+1)}{2}$$

independent components.

So, if we take a set of Killing vectors $\{\xi^{(n)}_\mu\}$, the initial values $\xi^{(n)}_\lambda(X)$ and $\xi^{(n)}_{\lambda;\nu}(X)$ should be thought of as the components of vectors in an $N(N+1)/2$-dimensional vector space. By definition, a vector space in d-dimensional because it has, at most, d linearly-independent vectors, so the maximum number of linearly-independent Killing vectors is $N(N+1)/2$. A space that has $N(N+1)/2$ Killing vectors for its metric is known as a **maximally symmetric space**.

The study of maximally symmetric spaces is particularly important in cosmology, specifically in the study of **homogeneous and isotropic spaces**. A space is **homogeneous** if for any two points p, q in the space, there exists an isometry of the metric that takes p to q, i.e. the Killing vector ξ_λ must be an arbitrary value. If at every point p in a space, we define u to be a time-like tangent vector, and s_1 and s_2 to be orthogonal space-like tangent vectors, and there exists an isometry of the metric such that p and u are unchanged, but s_1 is rotated into s_2, then the space is **isotropic** at every point. This requires that $\xi_\lambda = 0$, so as not to translate the point, but the derivative $\xi_{\lambda;\nu}$ must be an arbitrary matrix (which is anti-symmetric, due to the Killing equation). A space which is homogeneous and isotropic at one point is isotropic at every point, so these spaces are typically referred to simply as homogeneous and isotropic spaces.

For an N-dimensional space, it is clear that the homogeneity requirement defines N translational isometries. Further, there will be N different time-like vectors about which to rotate in an isotropic space, so that leaves $N-1$ space-like vectors to physically rotate (e.g. rotating s_1). To avoid double counting, there are $N(N-1)/2$ independent isometric rotations in an isotropic space. So, a space which is both homogeneous and isotropic must have $N(N+1)/2$ isometries, and is therefore maximally symmetric.

In order for a metric to admit the maximal number of Killing vectors depends on how many Killing vectors can be constructed out of their initial data points using a Taylor expansion, which we should recall is only possible in the form we've considered if equation (2.4) is integrable.

We can borrow another covariant derivative identity (a la Dirac (13.1)):

$$\xi_{\rho;\nu;\mu;\sigma} - \xi_{\rho;\nu;\sigma;\mu} = -R^\lambda_{\rho\mu\sigma}\xi_{\lambda;\nu} - R^\lambda_{\nu\mu\sigma}\xi_{\rho;\lambda} \tag{3.1}$$

Taking the covariant derivative of (2.4), we see that:

$$\xi_{\rho;\nu;\mu;\sigma} = -R^\lambda_{\mu\nu\rho;\sigma}\xi_\lambda - R^\lambda_{\mu\nu\rho}\xi_{\lambda;\sigma}$$

Likewise, by swapping indices in equation (2.4) and taking a covariant derivative, as we did above, we have:

$$\xi_{\rho;\nu;\sigma;\mu} = -R^\lambda_{\sigma\nu\rho;\mu}\xi_\lambda - R^\lambda_{\sigma\nu\rho}\xi_{\lambda;\mu}$$

Plugging these into equation (3.1):

$$\begin{aligned}
&\xi_{\rho;\nu;\mu;\sigma} - \xi_{\rho;\nu;\sigma;\mu} = -R^{\lambda}_{\rho\mu\sigma}\xi_{\lambda;\nu} - R^{\lambda}_{\nu\mu\sigma}\xi_{\rho;\lambda} \\
\Rightarrow\ &(-R^{\lambda}_{\mu\nu\rho;\sigma}\xi_{\lambda} - R^{\lambda}_{\mu\nu\rho}\xi_{\lambda;\sigma}) - (-R^{\lambda}_{\sigma\nu\rho;\mu}\xi_{\lambda} - R^{\lambda}_{\sigma\nu\rho}\xi_{\lambda;\mu}) = -R^{\lambda}_{\rho\mu\sigma}\xi_{\lambda;\nu} - R^{\lambda}_{\nu\mu\sigma}\xi_{\rho;\lambda} \\
\Rightarrow & - R^{\lambda}_{\mu\nu\rho}\xi_{\lambda;\sigma} + R^{\lambda}_{\sigma\nu\rho}\xi_{\lambda;\mu} + (R^{\lambda}_{\sigma\nu\rho;\mu} - R^{\lambda}_{\mu\nu\rho;\sigma})\xi_{\lambda} = -R^{\lambda}_{\rho\mu\sigma}\xi_{\lambda;\nu} - R^{\lambda}_{\nu\mu\sigma}\xi_{\rho;\lambda}
\end{aligned}$$

In the above equation, we can group by the Killing vector ξ_λ and by it derivative. Of the four terms that are derivatives, three of them are of the form $\xi_{\lambda;j}$, where j is an arbitrary index. The one outlier is the term $\xi_{\rho;\lambda}$, but due to the Killing equation, this is equal to $-\xi_{\lambda;\rho}$. So the above equation becomes:

$$R^{\lambda}_{\mu\nu\rho}\xi_{\lambda;\sigma} - R^{\lambda}_{\sigma\nu\rho}\xi_{\lambda;\mu} - R^{\lambda}_{\rho\mu\sigma}\xi_{\lambda;\nu} + R^{\lambda}_{\nu\mu\sigma}\xi_{\lambda;\rho} = (R^{\lambda}_{\sigma\nu\rho;\mu} - R^{\lambda}_{\mu\nu\rho;\sigma})\xi_{\lambda}$$

A $\xi_{\lambda;\kappa}$ can be factored from each term on the left-hand-side by inserting an appropriate Kronecker-delta in each term, causing us to arrive at:

$$(R^{\lambda}_{\mu\nu\rho}\delta^{\kappa}_{\sigma} - R^{\lambda}_{\sigma\nu\rho}\delta^{\kappa}_{\mu} - R^{\lambda}_{\rho\mu\sigma}\delta^{\kappa}_{\nu} + R^{\lambda}_{\nu\mu\sigma}\delta^{\kappa}_{\rho})\xi_{\lambda;\kappa} = (R^{\lambda}_{\sigma\nu\rho;\mu} - R^{\lambda}_{\mu\nu\rho;\sigma})\xi_{\lambda} \tag{3.2}$$

This is yet another requirement for Killing vectors, but we have yet to tie it into maximally symmetric spaces. We know that in a maximally symmetric space, we can find Killing vectors for which $\xi_\lambda(x) = 0$ and $\xi_{\lambda;\kappa}(x)$ is an arbitrary, anti-symmetric matrix (from the condition of isotropy). This means that the left-hand-side of the above equation must be identically equal to zero. A sure-fire way to ensure this is to require that the entire coefficient of $\xi_{\lambda;\kappa}$ remain unchanged under an exchange of λ and κ. Since the derivative of the Killing vector is antisymmetric, if the coefficient remains the same under a symmetry operation, it must be zero, satisfying the right-hand-side of the above equation when $\xi_\lambda(x) = 0$. Thus:

$$R^{\lambda}_{\mu\nu\rho}\delta^{\kappa}_{\sigma} - R^{\lambda}_{\sigma\nu\rho}\delta^{\kappa}_{\mu} - R^{\lambda}_{\rho\mu\sigma}\delta^{\kappa}_{\nu} + R^{\lambda}_{\nu\mu\sigma}\delta^{\kappa}_{\rho} = R^{\kappa}_{\mu\nu\rho}\delta^{\lambda}_{\sigma} - R^{\kappa}_{\sigma\nu\rho}\delta^{\lambda}_{\mu} - R^{\kappa}_{\rho\mu\sigma}\delta^{\lambda}_{\nu} + R^{\kappa}_{\nu\mu\sigma}\delta^{\lambda}_{\rho} \tag{3.3}$$

The above only considers Killing vectors for an isotropic space. For a homogeneous space, we know there must also be Killing vectors which are arbitrary. Keeping the above requirement for the isotropic Killing vectors means that the right-hand-side of equation (3.2) identically equal to zero as well, so it must be true that:

$$R^{\lambda}_{\sigma\nu\rho;\mu} = R^{\lambda}_{\mu\nu\rho;\sigma} \tag{3.4}$$

With the previous two conditions satisfied, one for isotropy and one for homogeneity, we guaranteed that our space is homogeneous and isotropic, and therefore admits $N(N+1)/2$ Killing vectors and is thus maximally symmetric. What we want to do now is use the above results to determine what the Riemann curvature tensor should be for a maximally symmetric space. Taking equation (3.3), we want to contract the κ and σ indices:

$$R^{\lambda}_{\mu\nu\rho}\delta^{\sigma}_{\sigma} - R^{\lambda}_{\sigma\nu\rho}\delta^{\sigma}_{\mu} - R^{\lambda}_{\rho\mu\sigma}\delta^{\sigma}_{\nu} + R^{\lambda}_{\nu\mu\sigma}\delta^{\sigma}_{\rho} = R^{\sigma}_{\mu\nu\rho}\delta^{\lambda}_{\sigma} - R^{\sigma}_{\sigma\nu\rho}\delta^{\lambda}_{\mu} - R^{\sigma}_{\rho\mu\sigma}\delta^{\lambda}_{\nu} + R^{\sigma}_{\nu\mu\sigma}\delta^{\lambda}_{\rho}$$

The sum of δ^{σ}_{σ} is just the dimensionality of the space, which we will assume is N. Note that because of the anti-symmetry conditions of the Riemann curvature tensor, the sum $R^{\sigma}_{\sigma\nu\rho}$ equals 0, and that the sum $R^{\sigma}_{\rho\mu\sigma}$ is, by definition, the Ricci tensor $R_{\rho\mu}$. So, the above equation reduces to:

$$NR^{\lambda}_{\mu\nu\rho} - R^{\lambda}_{\mu\nu\rho} - R^{\lambda}_{\rho\mu\nu} + R^{\lambda}_{\nu\mu\rho} = R^{\lambda}_{\mu\nu\rho} - R_{\rho\mu}\delta^{\lambda}_{\nu} + R_{\nu\mu}\delta^{\lambda}_{\rho} \tag{3.5}$$

Now we want to apply the first Bianchi identity to the second, third, and fourth terms on the left-hand-side, which states that:

$$R^{\lambda}_{\mu\nu\rho} + R^{\lambda}_{\rho\mu\nu} + R^{\lambda}_{\nu\rho\mu} = 0$$

Due to the antisymmetry of the Riemann curvature tensor, the third term is equal to $R^{\lambda}_{\nu\rho\mu} = -R^{\lambda}_{\nu\mu\rho}$, and thus we see that the above Bianchi identity means that the three terms identified previously in equation (3.5) sum to zero. So, we have, after grouping the factors of $R^{\lambda}_{\mu\nu\rho}$ on the left-hand-side of the equation:

$$(N-1)R^{\lambda}_{\mu\nu\rho} = -R_{\rho\mu}\delta^{\lambda}_{\nu} + R_{\nu\mu}\delta^{\lambda}_{\rho}$$

And, if we multiply both sides by $g_{\kappa\lambda}$ and sum, we can lower the indices of all the tensors:

$$(N-1)R_{\kappa\mu\nu\rho} = -R_{\rho\mu}g_{\kappa\nu} + R_{\nu\mu}g_{\kappa\rho} \tag{3.6}$$

Since the Riemann curvature tensor on the left-hand-side must by anti-symmetric about κ and μ, if we were to swap those indices on the right-hand-side of the equation, we'd pick up a negative sign due to this antisymmetry, so:

$$-R_{\rho\mu}g_{\kappa\nu} + R_{\nu\mu}g_{\kappa\rho} = +R_{\rho\kappa}g_{\mu\nu} - R_{\nu\kappa}g_{\mu\rho}$$

Multiplying by $g^{\kappa\rho}$ and summing yields:

$$\begin{aligned} &- R_{\rho\mu}\underbrace{g_{\kappa\nu}g^{\kappa\rho}}_{\delta^{\rho}_{\nu}} + R_{\nu\mu}\underbrace{g_{\kappa\rho}g^{\kappa\rho}}_{N} = \underbrace{R_{\rho\kappa}g^{\kappa\rho}}_{R^{\rho}_{\rho}}g_{\mu\nu} - R_{\nu\kappa}\underbrace{g_{\mu\rho}g^{\kappa\rho}}_{\delta^{\kappa}_{\mu}} \\ \Rightarrow \quad &- R_{\nu\mu} + NR_{\nu\mu} = Rg_{\mu\nu} - R_{\nu\mu} \end{aligned}$$

where the Ricci scalar is defined as $R = R^{\rho}_{\rho}$. Since the Ricci curvature tensor is symmetric, we can also say $R_{\nu\mu} = R_{\mu\nu}$ for cohesiveness. Thus, our result is:

$$R_{\mu\nu} = \frac{R}{N}g_{\mu\nu} \tag{3.7}$$

Plugging this result back into equation (3.6), we get:

$$(N-1)R_{\kappa\mu\nu\rho} = -R_{\rho\mu}g_{\kappa\nu} + R_{\nu\mu}g_{\kappa\rho} = -\frac{R}{N}g_{\rho\mu}g_{\kappa\nu} + \frac{R}{N}g_{\mu\nu}g_{\kappa\rho} = \frac{R}{N}(g_{\mu\nu}g_{\kappa\rho} - g_{\rho\mu}g_{\kappa\nu})$$

Thus, we arrive at the Riemann curvature tensor that describes a maximally symmetric space:

$$R_{\kappa\mu\nu\rho} = \frac{R}{N(N-1)}(g_{\mu\nu}g_{\kappa\rho} - g_{\rho\mu}g_{\kappa\nu}) \tag{3.8}$$

We set out to apply as many restrictions on the form of the Riemann curvature tensor as we could for a maximally symmetric space, and we've done just that; there's nothing else we can do to the above equation given the Killing vectors.

However, there is more to learn about the Ricci curvature tensor and the Ricci scalar. Using a well-known identity (a la Weinberg (6.8.4) or Dirac (14.3)):

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{;\mu} = 0$$

we can multiply a factor of $g_{\nu\rho}$ into the covariant derivative, resulting in:

$$\left(R^{\mu}_{\rho} - \frac{1}{2}\delta^{\mu}_{\rho}R\right)_{;\mu} = 0$$

Using equation (3.7), this means that:

$$R^{\mu}_{\rho} = \frac{R}{N}\delta^{\mu}_{\rho}$$

and so the above equation becomes:

$$\left(\frac{1}{N}\delta^{\mu}_{\rho}R - \frac{1}{2}g^{\mu}_{\rho}R\right)_{;\mu} = 0$$

or:

$$\left(\frac{1}{N} - \frac{1}{2}\right)R_{;\mu} = 0$$

Since R is a scalar, the covariant derivative is equal to the typical gradient, so the above equation is equivalent to:

$$\left(\frac{1}{N} - \frac{1}{2}\right)\frac{\partial R}{\partial x^{\mu}} = 0 \tag{3.9}$$

The result of this is that, for a space with dimension $N > 2$, the Ricci scalar R is a constant; that is, maximally symmetric spaces (with dimension greater than 2) have a **constant Ricci curvature**.

In equation (3.8), we had a term $R/N(N-1)$. Because this is cumbersome, it is convenient to introduce a different curvature scalar K such that:

$$K = \frac{R}{N(N-1)}$$

This means that equation (3.8) can be re-written as:

$$R_{\kappa\mu\nu\rho} = K(g_{\mu\nu}g_{\kappa\rho} - g_{\rho\mu}g_{\kappa\nu}) \tag{3.10}$$

where K is a constant since R is a constant.

There is a very important theorem that Weinberg proves, but I won't prove here, which I will call the **theorem of metric uniqueness**. From Weinberg,

> *Given two maximally symmetric metrics with the same K and the same signature, it will always be possible to find a coordinate transformation that carries one metric into another.*

i.e. that a metric is uniquely defined, up to a coordinate transformation, for a given curvature K and signature.

The last thing to do now is to take all that we have gathered about maximally symmetric spaces, which bear in mind came entirely from the Killing equation and from the requirements on the Killing vectors that homogeneity and isotropy demand, we can construct the metrics for any maximally symmetric space. It turns out that it won't be necessary to consider any arbitrary curvature K, because the curvature can easily be rescaled in the metric; it will only be necessary to consider spaces of constant positive curvature, zero curvature, and constant negative curvature. Spaces of constant positive curvature are **spherical**, spaces of zero curvature are **flat**, and spaces of constant negative curvature are **hyperbolic**, so these are the only three spaces (with a regular topology; nothing strange has been considered here) that are maximally symmetric. The end result of this last derivation will be the Robertson-Walker metric, which is the metric used in cosmology.

Note that all of the work up to this point has, essentially, been to rigorously derive the single equation for the Riemann curvature tensor for a maximally symmetric space. This is *not* the most common way, by any means; in fact, it is incredibly uncommon, and Weinberg is the only (popular) reference that I found with such a detailed derivation. Carroll has a much more popular derivation for the same equation, which uses a less rigorous approach to exploiting the symmetry, but arrives at the same result in almost no time at all.

The basic argument is this: if space is homogeneous and isotropic, then your Riemann curvature tensor should be invariant under any (local) Lorentz transformation (i.e. a local change of basis should leave the Riemann curvature tensor invariant). There are only three unique tensors that have this property, though: the metric tensor, the Kronecker-delta, and the Levi-Civita tensor. This means that the Riemann curvature tensor must be constructed out of some linear combination of these three tensors, while still maintaining the appropriate anti-symmetry relations. By brute force, it is possible to show that there is only one such combination:

$$R_{\kappa\mu\nu\rho} \propto g_{\mu\nu}g_{\kappa\rho} - g_{\rho\mu}g_{\kappa\nu}$$

Setting the proportionality constant equal to some c and contracting over all indices, the left-hand-side becomes R, the Ricci curvature, and the right-hand-side becomes $cN(N-1)$, so we would arrive at the same equation for the Riemann curvature tensor:

$$R_{\kappa\mu\nu\rho} = \frac{R}{N(N-1)}(g_{\mu\nu}g_{\kappa\rho} - g_{\rho\mu}g_{\kappa\nu}) = K(g_{\mu\nu}g_{\kappa\rho} - g_{\rho\mu}g_{\kappa\nu})$$