Derivation of the Robertson-Walker Metric

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1 Introduction

The Robertson-Walker metric is the most general metric for the universe that agrees with the cosmological principle, which is the guiding principle in all of cosmology and seems to be a fairly accurate statement as far as observations are concerned. The cosmological principle states, simply, that the universe must look the same to all observers within it, which requires that the universe be homogeneous and isotropic everywhere. A homogeneous universe is one that looks the same at every location, x, and an isotropic universe is one that looks the same in every direction, (θ, ϕ). Though the universe doesn't seem at all homogeneous (look at how scattered mass is in our solar system, galaxy, galactic neighborhood, etc.), at *cosmological distances*, typically about 1 Gly or greater, the universe appears to be extremely homogeneous, according to deep quasar studies performed recently, as shown in Figure 1.

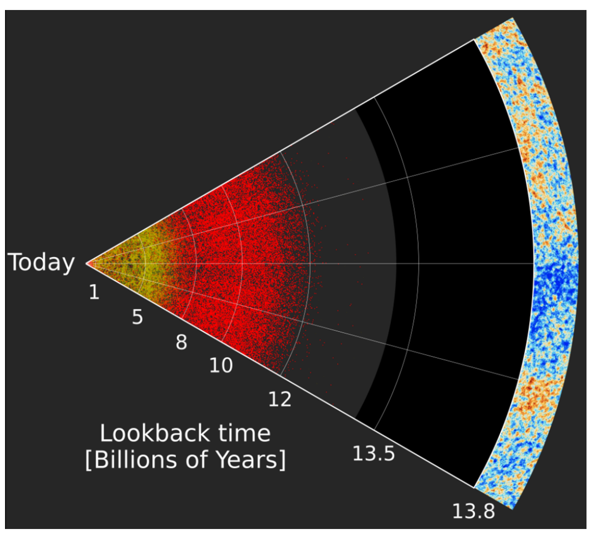

Figure 1: Deep quasar study showing homogeneity at cosmological distances. *Image credit: Sloan Digital Sky Survey.*

While the deep quasar study showed good agreement with homogeneity in the universe, the cosmic microwave background (CMB) has shown extremely good agreement with isotropy in the universe, with the so-called an-isotropies being on the order of 1 part per 10^5. The CMB is a near-perfect blackbody spectrum, with a temperature measured at $T = 2.7255 \pm 0.0000$ K[1]. The current-generate satellite measuring the CMB is the Planck spacecraft, replacing the Wilkinson Microwave

[1] C. Patrignani et al., "Particle Data Group", *Chin. Phys. C*, **40**, 100001 (2016).

Anisotropy Probe (WMAP) satellite. Extremely accurate measurements on various cosmological parameters are being taken by Planck based on measuring the anisotropies with extremely small angular resolution.

With general acceptance of the cosmological principle comes the acceptance of only three possible geometries of the universe: flat, spherical, or hyperbolic, which are the only three possible geometries of a universe equipped with the Robertson-Walker metric. To very high accuracy, the universe has been measured to be very flat, which presents an issue for the classical big model, for reasons I won't get into[2]. The universe is also measured to be expanding, which is another consequence of the universe being equipped with the Robertson-Walker metric (and the current energy distribution of the universe, which won't be discussed in this note).

This note utilizes the Riemann curvature tensor for general maximally symmetric (sub)space to derive the Robertson-Walker metric. Refer to my note on maximally symmetric spaces for the derivation of the Riemann curvature tensor that I will use. Further, this note will use a general form of a static, asymptotically flat 3-space:

$$d\sigma^2 = e^{2\beta(r)}dr^2 + r^2 d\Omega^2$$

This metric is well-known, as it is typically (a part of) the starting point in deriving the famous Schwarzchild metric. For instance, see equation (5.5) in Carroll[3].

2 The Derivation

To obey the cosmological principle, 3-space must be homogeneous and isotropic, i.e. must be a maximally symmetric manifold. We don't expect the complete 4-spacetime metric to include any $dtdx^i$ terms (for $i = 1, 2, 3$), because these only appear in the presence of strong gravitational fields (e.g. near a black hole), and we don't want g_{tt} to depend on r because this would mean various observers around the universe would measure time differently. The most general 4-spacetime metric we can use is then:

$$ds^2 = -dt^2 + a^2(t)d\sigma^2 \tag{1}$$

where $d\sigma^2$ is the maximally symmetric 3-space metric, and $a(t)$ is known as the scale-factor. Once again, the scale-factor cannot depend on r, because then various observers across the universe would measure a different scale factor, violating homogeneity.

We can define the 3-space metric generally as:

$$d\sigma^2 = \gamma_{ij}du^i du^j \tag{2}$$

where the coordinates u are the coordinates in 3-space, and the indices i, j run from 1 to 3. The metric γ_{ij} is going to be maximally symmetric, and therefore must describe a Riemann curvature tensor of the form:

$$\tilde{R}_{ijkl} = K(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) \tag{3}$$

where K is the Gaussian curvature of 3-space, and I'm using the tilde to denote that $\tilde{R}$ is the 3-space Riemann curvature tensor, as opposed to the 4-spacetime Riemann curvature tensor $R_{\mu\nu\rho\sigma}$. Since 4-spacetime doesn't need to be homogeneous and isotropic (the cosmological principle is a requirement of 3-space), $R_{\mu\nu\rho\sigma}$ doesn't need to follow the general form given above.

[2] This is known as the flatness problem of the classical big bang, and the modern theory of cosmic inflation seems to correct it, along with some other problems with the classical big bang, very nicely.

[3] S. Carroll, "Spacetime & Geometry: an Introduction to General Relativity," *Addison Wesley* (2004).

We can contract two of the indices on $\tilde{R}_{ijkl}$ by raising i and then summing over $i = k = a$, yielding the (3-space) Ricci tensor $\tilde{R}_{jk}$:

$$\gamma^{ia}\tilde{R}_{ijkl} = \tilde{R}^a_{jkl} = K(\gamma^{ia}\gamma_{ik}\gamma_{jl} - \gamma^{ia}\gamma_{il}\gamma_{jk}) = (\gamma^a_k\gamma_{jl} - \gamma^a_l\gamma_{jk})$$

Setting $a = k$:

$$R^a_{jal} = R_{jl} = K(\underbrace{\gamma^a_a}_{=3}\gamma_{jl} - \underbrace{\gamma^a_l\gamma_{ja}}_{=\gamma_{jl}})$$

Thus, the Ricci tensor is:

$$R_{jl} = 2K\gamma_{jl} \tag{4}$$

The general 3-space metric for a spherically-symmetric 3-space is going to be given by:

$$d\sigma^2 = e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \tag{5}$$

which is the most general form of the metric that will agree with the Riemann curvature tensor or a maximally symmetric 3-space. The non-zero (3-space) Christoffel symbols of this metric are well-known[4]:

$$\begin{aligned} &\tilde{\Gamma}^r_{rr} = \frac{d\beta}{dr} && \tilde{\Gamma}^\theta_{r\theta} = \frac{1}{r} && \tilde{\Gamma}^r_{\theta\theta} = -re^{-2\beta} \\ &\tilde{\Gamma}^\phi_{r\phi} = \frac{1}{r} && \tilde{\Gamma}^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta && \tilde{\Gamma}^\theta_{\phi\phi} = -\sin\theta\cos\theta \\ &\tilde{\Gamma}^\phi_{\theta\phi} = \cot\theta \end{aligned} \tag{6}$$

All other non-zero components of the Christoffel symbols can be found using the symmetry relation $\tilde{\Gamma}^k_{ij} = \tilde{\Gamma}^k_{ji}$. From the Christoffel symbols, we can compute the (3-space) Ricci tensor components[5]:

$$\begin{aligned} \tilde{R}_{rr} &= \frac{2}{r}\frac{d\beta}{dr} \\ \tilde{R}_{\theta\theta} &= e^{-2\beta}\left(r\frac{d\beta}{dr} - 1\right) + 1 \\ \tilde{R}_{\phi\phi} &= \left[e^{-2\beta}\left(r\frac{d\beta}{dr} - 1\right) + 1\right]\sin^2\theta \end{aligned} \tag{7}$$

Using equation (4), we can solve for $e^{-2\beta}$ using the rr equation:

$$\frac{2}{r}\frac{d\beta}{dr} = 2Ke^{2\beta} \quad \Rightarrow \quad \int_0^\beta e^{-2\beta'}d\beta' = \int_0^r Kr'dr' \quad \Rightarrow \quad -\frac{1}{2}\left(e^{-2\beta} - 1\right) = \frac{1}{2}Kr^2$$

$$\Rightarrow e^{-2\beta} = 1 - Kr^2$$

Taking the inverse, we have:

$$e^{2\beta(r)} = \frac{1}{1 - Kr^2} \tag{8}$$

and, thus, our 3-space metric, given by equation (5), is:

$$d\sigma^2 = \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \tag{9}$$

[4]See Carroll, equation (5.12), for instance.

[5]See Carroll, equation (5.14), for example.

This metric depends on on the exact curvature K, but only upon inspection; the metric can actually be reparametrized such that any curvature K can be normalized to $+1$ or -1 (if it's zero, nothing needs to be done, obviously). Define any curvature K such that:

$$K = \alpha k \qquad k \in (-1, 0, +1) \tag{10}$$

We can reparametrize the radial coordinate r such that:

$$\bar{r} = \sqrt{\alpha} r \tag{11}$$

in which case the metric becomes:

$$d\sigma^2 = \frac{1}{\alpha}\frac{d\bar{r}^2}{1 - k\bar{r}^2} + \frac{1}{\alpha}\bar{r}^2 d\Omega^2$$

All we need to do is pull the factor of $1/\alpha$ out of $d\sigma^2$ so that it can be absorbed into the scale factor $a(t)$; then, our 3-space metric is:

$$d\bar{\sigma}^2 = \frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2 d\Omega^2$$

and therefore we only ever have to consider the cases of $k = -1, 0, +1$. Because these are the only cases we'll ever consider, it's easier to call the radial coordinate such that k is normalized as r, instead of $\bar{r}$ as I just did. Then, the metric is:

$$d\sigma^2 = \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \tag{12}$$

Given the solution for the 3-space metric, there are two common ways of giving the complete Robertson-Walker for 4-spacetime, depending on how you want to interpret the scale-factor. Some prefer to keep it as a unitless factor that takes 1 meter at todays time (defined to be $t = 0$) and tells you how 1 meter would relate at any time in the past ($t < 0$) or any time in the future ($t > 0$). In this case, we simply plug $d\sigma^2$ into our full 4-spacetime metric in equation (1) as-is:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right) \tag{13}$$

On the other hand, many people prefer to give the scale factor the units of distance, meaning that the radial coordinate r measures a sort of unitless lattice distance between two objects in the universe that is independent of time. The scale factor with units, written as $R(t)$, then tells you how far, physically, two lattice points are at time in the past or future. In this case, the full 4-spacetime Robertson-Walker metric is:

$$ds^2 = -dt^2 + R^2(t)\left(\frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2 d\Omega^2\right) \tag{14}$$

where I'm once again using the bar notation for the radial coordinate, but this time to signify that $\bar{r}$ is unitless. This is a fairly common notation, and it's the one I prefer, though notation varies wildly in general relativity with the source material.

3 Flat, Spherical, and Hyperbolic Universes

The last thing to do is prove that the three geometries given by the Robertson-Walker metric, corresponding to the three choices $k = -1, 0, +1$, yield hyperbolic, flat, and spherical universes, respectively. Instead of proving this in the "forward" direction in which I just stated it, I'll prove this in the "reverse" direction (the statement is a biconditional, so it doesn't matter which direction you prove it; they should both be correct). In this case, I want to embed 3-dimensional sphere and a 3-dimensional hyperbola into a 4-dimensional flat background space, and prove that those embeddings correspond to the $k = +1$ and $k = -1$ cases, respectively. The fact that the $k = 0$ case is a flat universe is clear upon inspection:

$$d\sigma^2 = dr^2 + r^2 d\Omega^2 \tag{15}$$

which is just a flat 3-space in spherical coordinates.

Consider a unit 3-sphere, given by the equation:

$$x^2 + y^2 + z^2 + w^2 = 1 \tag{16}$$

We want to embed this in a flat, 4-space background, which would be governed by the metric:

$$d\sigma^2 = dx^2 + dy^2 + dz^2 + dw^2$$

Note that this embedding results in a 3-space, like we want, since the constraint equation of points in the space to be on the unit 3-sphere will allow us to eliminate w as a coordinate, leaving only 3 coordinates remaining. Taking the derivative of the constraint equation:

$$2xdx + 2ydy + 2zdz + 2wdw = 0 \qquad \Rightarrow \qquad dw = -\frac{xdx + ydy + zdz}{w}$$

$$\Rightarrow \qquad dw^2 = \frac{(xdx + ydy + zdz)^2}{1 - x^2 - y^2 - z^2}$$

So, the metric of the embedded unit 3-sphere is:

$$d\sigma^2 = dx^2 + dy^2 + dz^2 + \frac{(xdx + ydy + zdz)^2}{1 - x^2 - y^2 - z^2} \tag{17}$$

To complete this embedding, we need to convert from Cartesian to spherical coordinates. In the embedding space, which is 4-dimensional, spherical coordinates will be given by (r, ψ, θ, ϕ), where r, θ, and ϕ all play identical roles in the 4-space as in the 3-space, but there is a new polar angle ψ which defines the projection of $\mathbf{r}$ in the 4-space into a 3-subspace, much like θ allows for the projection from a sphere onto a plane. In terms of these spherical coordinates, the Cartesian coordinates are what we'd expect:

$$\begin{aligned} x &= \sin\psi \sin\theta \cos\phi \qquad & y &= \sin\psi \sin\theta \sin\phi \\ z &= \sin\psi \cos\theta \qquad & w &= r \cos\psi \end{aligned} \tag{18}$$

Note that no factors of r appear in the Cartesian-to-spherical coordinate transformations, because on the unit-sphere, $r = 1$. So, the derivatives are:

$$dx = \cos\psi \sin\theta \cos\phi d\psi + \sin\psi \cos\theta \cos\phi d\theta - \sin\psi \sin\theta \sin\phi d\phi$$

$$dy = \cos\psi \sin\theta \sin\phi d\psi + \sin\psi \cos\theta \sin\phi d\theta + \sin\psi \sin\theta \cos\phi d\phi$$

$$dz = \cos\psi \cos\theta d\psi - \sin\psi \sin\theta d\theta$$

So, $xdx + ydy + zdz$ is:

$$xdx + ydy + zdz = \sin\psi\cos\psi d\psi$$

Notice that, in the 4-space, $1 - x^2 - y^2 - z^2 = w^2 = \cos^2\psi$, so we have one of the two terms from the metric for our 3-sphere:

$$\frac{(xdx + ydy + zdz)^2}{1 - x^2 - y^2 - z^2} = \sin^2\psi d\psi^2 \tag{19}$$

The other term, $dx^2 + dy^2 + dz^2$, takes considerably more algebra to compute. Roughly, the squares are given by:

$$\begin{aligned}
dx^2 &= \cos^2\psi\sin^2\theta\cos^2\phi d\psi^2 + \sin^2\psi\cos^2\theta\cos^2\phi d\theta^2 + \sin^2\psi\sin^2\theta\sin^2\phi d\phi^2 \\
&\quad + 2\sin\psi\cos\psi\sin\theta\cos\theta\cos^2\phi d\psi d\theta + 2 \text{ terms which cancel} \\
dy^2 &= \cos^2\psi\sin^2\theta\sin^2\phi d\psi^2 + \sin^2\psi\cos^2\theta\sin^2\phi d\theta^2 + \sin^2\psi\sin^2\theta\cos^2\phi d\phi^2 \\
&\quad + 2\sin\psi\cos\psi\sin\theta\cos\theta\sin^2\phi d\psi d\theta + 2 \text{ terms which cancel} \\
dz^2 &= \cos^2\psi\cos^2\theta d\psi^2 + \sin^2\psi\sin^2\theta d\theta^2 - 2\sin\psi\cos\psi\sin\theta\cos\theta d\psi d\theta
\end{aligned}$$

So, the sum of these square terms is, after cancelling and summing any $\sin^2 x + \cos^2 x = 1$, the first term from our 3-sphere metric:

$$dx^2 + dy^2 + dz^2 = \cos^2\psi d\psi^2 + \sin^2\psi d\theta^2 + \sin^2\psi\sin^2\theta d\phi^2 \tag{20}$$

Combining equations (19) and (20) give us the 3-space metric of a unit 3-sphere:

$$d\sigma^2 = d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)$$

Using the definition of $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ as the metric of a unit 2-sphere, we simply re-write our metric as:

$$d\sigma^2 = d\psi^2 + \sin^2\psi d\Omega^2 \tag{21}$$

The question now is: is this the same metric as the (3-space) Robertson-Walker metric, given by equation (12), for $k = 1$? First, as a side note, what if we had embedded an arbitrary 3-sphere instead of a 4-sphere? If it had a radius r, i.e. the equation of constraint was $x^2 + y^2 + z^2 + w^2 = r^2$, then the metric would end up with a factor of r^2 at the front. This factor could easily be absorbed into the scale factor $a(t)$, leaving us with the above form of the metric no matter what sphere we choose to embed. Let's define a radial coordinate $\bar{r}$ such that:

$$\bar{r} = \sin\psi \tag{22}$$

Note that since ψ is an angle, the above metric is best cast if we take the units over from $d\sigma^2$ to $a(t)$, though as of right now, $d\sigma^2$ technically carries the units because our coordinate transformations all had an implicit factor of 1m multiplied. But we'll work with the most physically intuitive form, which in this case is to leave $d\sigma^2$ unitless (so we'd call it $d\bar{\sigma}^2$, actually) and therefore $\bar{r}$ is actually a measure of angle. Taking the derivative of $\bar{r}$:

$$d\bar{r} = \cos\psi d\psi \quad \Rightarrow \quad d\bar{r}^2 = \cos^2\psi d\psi^2 = (1 - \sin^2\psi)d\psi^2$$

Since $\bar{r} = \sin\psi$, we can replace all factors of ψ with their corresponding factors of $\bar{r}$ in $d\bar{\sigma}^2$, yielding:

$$d\bar{\sigma}^2 = \frac{d\bar{r}^2}{1 - \bar{r}^2} + \bar{r}^2 d\Omega^2 \tag{23}$$

which is exactly the (3-space) Robertson-Walker metric, equation (12), for $k = +1$.

Because of the tediousness of the algebra, I won't prove the embedding of the unit 3-hyperbola into a flat background 4-space, but the process is exactly the same: write down an equation of constraint for a unit 3-hyperbola (same as a unit 3-sphere with one negative sign), make the coordinate transformation from Cartesian to spherical coordinates, and simply the terms in $d\sigma^2$. The metric you would end with is:

$$d\bar{\sigma}^2 = d\psi^2 + \sinh^2\psi d\Omega^2 \tag{24}$$

This time, we make the substitution:

$$\bar{r} = \sinh\psi \tag{25}$$

in which case:

$$d\bar{r} = \cosh\psi d\psi \quad \Rightarrow \quad d\bar{r}^2 = \cosh^2\psi d\psi^2 = (1+\sinh^2\psi)d\psi^2$$

Thus, the 3-space metric describing the unit 3-hyperbola is:

$$d\bar{\sigma}^2 = \frac{d\bar{r}^2}{1+\bar{r}^2} + \bar{r}^2 d\Omega^2 \tag{26}$$

which is exactly the (3-space) Robertson-Walker metric for $k = -1$.

So, it has been shown that the most general solution for a maximally symmetric 3-space is the 3-space Robertson-Walker metric, given by equation (12). This then gives us the full 4-spacetime Robertson-Walker metric, which describes the most general universe obeying the cosmological principle. It was also shown that the Robertson-Walker metric corresponds to three geometries: flat, spherical, and hyperbolic. There is much more to study in cosmology, as this is literally the first step. The next thing to do is to derive the Friedmann equations, from which all of the physics of cosmology is derived.