Solutions to Gaussian Integrals

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The basic Gaussian integral is:

$$I = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx$$

Someone figured out a very clever trick to computing these integrals, and "higher-order" integrals of $x^n e^{-\alpha x^2}$. First, let's "square" this integral, in the sense:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-\alpha x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-\alpha y^2} dy \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\alpha(x^2+y^2)} dxdy \tag{1}$$

We want to keep each integral in terms of independent variables since the solution to each factor of I must be independent of the other.

Here comes the trick: we simply switch from Cartesian to polar coordinates. $dxdy$, each from $-\infty$ to ∞, describes an identical plane as $dA = rdrd\theta$, with r from 0 to ∞ and θ from 0 to 2π. With this change of variable comes the re-definition of distance in the plane as $x^2 + y^2 = r^2$. Since the plane is infinite, there's no difference in the circular plane and the rectangular plane. Plugging this substitution into I^2, equation (1) becomes:

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-\alpha r^2} rdrd\theta = 2\pi \int_0^{\infty} re^{-\alpha r^2} dr \tag{2}$$

This is why we want to make this change of variable: now we can solve the integral with simple u-substitution. Define:

$$u = \alpha r^2 \quad \Rightarrow \quad du = 2\alpha rdr$$

Then, equation (2) becomes:

$$I^2 = \frac{\pi}{\alpha} \int_0^{\infty} e^{-u} du = -\frac{\pi}{\alpha} \left[e^{-\infty} - e^0 \right] = \frac{\pi}{\alpha}$$

So, the solution to the integral I, which is our Gaussian integral, is just the square-root of the solution to the integral I^2, which is:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \tag{3}$$

Computing higher-order Gaussian integrals uses an equally clever trick. First of all, notice that all "odd-ordered" Gaussian integrals are zero:

$$\int_{-\infty}^{\infty} x^{2n+1} e^{-\alpha x^2} dx = 0 \tag{4}$$

This is because the Gaussian factor, $e^{-\alpha x^2}$, is an even function. Since x is an odd function, any power x^n where n is odd is an odd function, and thus so is $x^{2n+1}e^{-\alpha x^2}$. The integral of any odd function over the whole number line is always zero.

To find the "even-ordered" Gaussian integrals, we first notice the following:

$$-\frac{d}{d\alpha}e^{-\alpha x^2} = x^2 e^{-\alpha x^2}$$

which is exactly the function we're trying to integrate (at least for the second-order integral). In general, we would find that:

$$(-1)^n \frac{d^n}{d\alpha^n}e^{-\alpha x^2} = x^{2n}e^{-\alpha x^2} \tag{5}$$

We can then take our simple Gaussian integral, the "zeroth-order" Gaussian integral, and extend it to higher (even) order by noticing:

$$\int_{-\infty}^{\infty} x^{2n}e^{-\alpha x^2}dx = (-1)^n \int_{-\infty}^{\infty} \frac{d^n}{d\alpha^n}e^{-\alpha x^2}dx = (-1)^n \frac{d^n}{d\alpha^n}\left[\int_{-\infty}^{\infty} e^{-\alpha x^2}x\right]$$

Recalling the solution to the zeroth-order Gaussian integral, equation (4), we see that all (even) higher-order Gaussian integrals can be obtained with the formula:

$$\int_{-\infty}^{\infty} x^{2n}e^{-\alpha x^2}dx = (-1)^n \frac{d^n}{d\alpha^n}\left(\sqrt{\frac{\pi}{\alpha}}\right) \tag{6}$$

One can look up a general solution to the derivative on the right-hand-side, but almost always, in practice, it's only worth knowing up to the second-order Gaussian integral. This is found by taking the first derivative in terms of α:

$$\frac{d}{d\alpha}\left(\sqrt{\frac{\pi}{\alpha}}\right) = \frac{1}{2}\left(\sqrt{\frac{\pi}{\alpha}}\right)^{-1/2} * (-1)\frac{\pi}{\alpha^2} = -\frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

So, our second-order Gaussian integral is:

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}dx = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}} \tag{7}$$