Feynman's Guess at a Path Integral

Douglas H. Laurence

Department of Physical Sciences, Broward College, Davie, FL 33314

1 Introduction

Path integration is an essential method of quantizing classical field theories, especially used in the study of gauge field theories. During Richard Feynman's time as a graduate student at Princeton University, his attention was drawn to Paul Dirac's attempts to re-formulate quantum mechanics in terms of Lagrangian mechanics instead of Hamiltonian mechanics, as both Heisenberg's and Schrödinger's formulations were framed. Feynman claims, in his 1965 Nobel Lecture, that a visiting physicist, Herbert Jehle, pointed him towards Dirac's 1932 paper "The Lagrangian in Quantum Mechanics"[1] which proposed an "equivalence" between the classical idea of a propagator and the quantum idea of a propagator, in the form of:

$$K(x,t';x,t) \text{ is equivalent to } \exp\left[\frac{i\epsilon}{\hbar}L\right]$$

where K is the quantum propagator (a so-called "kernel") and L is the classical Lagrangian. Feynman was able to immediately, in front of Jehle, derive the Schrödinger equation from this idea by making two simple assumptions:

1. The "equivalence," which wasn't made clear by Dirac, was a proportionality
2. The Lagrangian was the simple, non-relativistic one: $\frac{1}{2}m\dot{x}^2 - V(x)$

The purpose of this note is to give this simple derivation, which is also outlined in Feynman's doctoral thesis[2].

2 Feynman's Simple Derivation of Schrödinger's Equation

As stated in the previous section, Feynman wanted to re-formulate quantum mechanics in terms of Lagrangian mechanics, which had been attempted by Dirac previously. Feynman proposed that a quantum propagator, $K(x,t';x,t)$ which takes a wave function from (x,t) to (x,t'), was simply proportional to the classical amplitude in terms of the Lagrangian, in the form of:

$$\psi(x,t+\epsilon) = A\int dy \exp\left[\frac{i\epsilon}{\hbar}L\left(\frac{x-y}{\epsilon},x\right)\right]\psi(y,t) \tag{1}$$

where the wavefunction $\psi(x,t+\epsilon)$, for a new time $t' = t+\epsilon$, on the right-hand-side is the propagated wavefunction, and the right-hand-side is the effect of the classical propagator on a wavefunction at

[1] P.A.M. Dirac, "The Lagrangian in Quantum Mechanics," *Phys. Z. Sowjetunion*, **3**, 64 (1932).
[2] L.M. Brown, "Feynman's Thesis: A New Approach to Quantum Theory," *World Scientific* (2005).

the initial time, $\psi(x,t)$. The Lagrangian used is the non-relativistic Lagrangian:

$$L\left(\frac{x-y}{\epsilon},x\right)=\frac{1}{2}m\left(\frac{x-y}{\epsilon}\right)^2-V(x) \tag{2}$$

where the factor $(x-y)/\epsilon$ is the approximation of $\dot{x}$ for a very small time interval ϵ.

The derivation of the Schrödinger equation from the above is very simple: Taylor expand both sides to some order in ϵ, set terms of the same order equal to one another, and see what you get. It turns out that the terms of order ϵ yield Schrödinger's equation. Before doing this, though, we want to make the substitution:

$$\eta = x-y$$

so that our integral equation becomes:

$$\psi(x,t+\epsilon)=A\int d\eta \exp\left[\frac{i\epsilon}{\hbar}L\left(\frac{\eta}{\epsilon},x\right)\right]\psi(x+\eta,t) \tag{3}$$

We're going to have three different Taylor expansions: that of $\psi(x,t+\epsilon)$ on the left-hand-side, and both $\exp\left[\frac{i\epsilon}{\hbar}L\left(\frac{\eta}{\epsilon},x\right)\right]$ and $\psi(x+\eta,t)$ on the right-hand-side. However, for reasons that will soon be clear, w don't want to Taylor expand *all* of $\exp\left[\frac{i\epsilon}{\hbar}L\left(\frac{\eta}{\epsilon},x\right)\right]$, but just the portion containing the potential $V(x)$. What I mean by this is we split the exponential like:

$$\exp\left[\frac{i\epsilon}{\hbar}L\left(\frac{\eta}{\epsilon},x\right)\right]=\exp\left[\frac{i\epsilon}{\hbar}\left(\frac{m\eta^2}{2\epsilon^2}-V(x)\right)\right]=\exp\left[\frac{i}{\hbar}\frac{m\eta^2}{2\epsilon}\right]\exp\left[-\frac{i\epsilon}{\hbar}V(x)\right]$$

and we only take the Taylor expansion of the term $\exp\left[-\frac{i\epsilon}{\hbar}V(x)\right]$. Performing these expansions:

$$\psi(x,t+\epsilon)=\psi(x,t)+\epsilon\frac{\partial\psi}{\partial t}(x,t)+...$$

$$\exp\left[-\frac{i\epsilon}{\hbar}V(x)\right]=1-\frac{i\epsilon}{\hbar}V(x)+...$$

$$\psi(x+\eta,t)=\psi(x,t)+\eta\frac{\partial\psi}{\partial x}(x,t)+\frac{1}{2}\eta^2\frac{\partial^2\psi}{\partial x^2}(x,t)+...$$

You may be put off by the fact that I expanded the last term to η^2, since I said we were performing each expansion to first order. But recall that I said first order in ϵ, with no mention of η. Ideally, η would be a small number, to aid in the validity of the expansion, but by definition of the integral, we have to consider some very large quantities of η (the integral runs from $-\infty$ to $+\infty$). The whole point of this isn't that it's completely valid, but that it was a *first attempt*, and a rather good one at that.

Let's focus on the right-hand-side of equation (3) exclusively for a little bit. First, let's multiply our two Taylor expansions together, keeping terms up to order ϵ:

$$\left[1-\frac{i\epsilon}{\hbar}V(x)+...\right]\left[\psi+\eta\frac{\partial\psi}{\partial x}+\eta^2\frac{\partial^2\psi}{\partial x^2}+...\right]$$

$$=\psi+\eta\frac{\partial\psi}{\partial x}+\frac{1}{2}\eta^2\frac{\partial^2\psi}{\partial x^2}-\frac{i\epsilon}{\hbar}V(x)\left(\psi+\eta\frac{\partial\psi}{\partial x}+\frac{1}{2}\eta^2\frac{\partial^2\psi}{\partial x^2}\right)+...$$

where I have dropped the explicit labels (x,t) on all wavefunctions (and their derivatives) since they are all defined at (x,t) after the Taylor expansions.

All of the above terms are going to be integrated from $-\infty$ to $+\infty$ over $d\eta$, weighted by the remaining piece of the exponential $\exp\left[\frac{i}{\hbar}\frac{m\eta^2}{2\epsilon}\right]$. Here is why this piece was left alone: all integrals become Gaussian integrals. Recall that the first three Gaussian integrals are:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

$$\int_{-\infty}^{\infty} x e^{-\alpha x^2} dx = 0$$

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = -\frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

I've included an appendix with the derivations of these formulas. Right away, we can conclude that all terms linear in η are going to vanish under integration, leaving terms of η^0 and η^2. So, if we integrate the right-hand-side of equation (3), which I will call $\mathcal{I}$, we have:

$$\mathcal{I} = A\int_{-\infty}^{\infty} d\eta \left(\psi + \eta^2 \frac{\partial^2\psi}{\partial x^2} - \frac{i\epsilon}{\hbar}V(x)\left(\psi + \frac{1}{2}\eta^2\frac{\partial^2\psi}{\partial x^2}\right) + ...\right) e^{\frac{im}{2\hbar\epsilon}\eta^2} \tag{4}$$

We, fundamentally, have two integrals here: the zeroth-order Gaussian and the second-order Gaussian, who's solutions are:

$$\int_{-\infty}^{\infty} d\eta e^{\frac{im}{2\hbar\epsilon}\eta^2} = \sqrt{-\frac{2\pi\hbar\epsilon}{im}} = \sqrt{\frac{2i\pi\hbar\epsilon}{m}}$$

$$\int_{-\infty}^{\infty} d\eta\eta^2 e^{\frac{im}{2\hbar\epsilon}\eta^2} = \frac{i\hbar\epsilon}{m}\sqrt{\frac{2i\pi\hbar\epsilon}{m}}$$

Plugging those solutions in, equation (4) results in:

$$\mathcal{I} = A\left(\sqrt{\frac{2i\pi\hbar\epsilon}{m}}\psi + \frac{i\hbar\epsilon}{2m}\sqrt{\frac{2i\pi\hbar\epsilon}{m}}\frac{\partial^2\psi}{\partial x^2} - \frac{i\epsilon}{\hbar}\sqrt{\frac{2i\pi\hbar\epsilon}{m}}V(x)\psi - \frac{i\epsilon}{\hbar}\frac{i\hbar\epsilon}{2m}\sqrt{\frac{2i\pi\hbar\epsilon}{m}}\frac{\partial^2\psi}{\partial x^2} + ...\right)$$

Notice that we can pull a common factor of $\sqrt{\frac{2i\pi\hbar\epsilon}{m}}$ out from each term, and factor by ψ and ψ'', so the right-hand-side of equation (3) becomes:

$$\mathcal{I} = A\sqrt{\frac{2i\pi\hbar\epsilon}{m}}\left[\left(1 - \frac{i\epsilon}{\hbar}V(x)\right)\psi + \frac{i\hbar\epsilon}{2m}\left(1 - \frac{i\epsilon}{\hbar}\right)\frac{\partial^2\psi}{\partial x^2} + ...\right]$$

Notice something, though: the second ψ'' term isn't linear in ϵ, but quadratic, so it must be dropped. Therefore, $\mathcal{I}$, to order ϵ, is actually:

$$\mathcal{I} = A\sqrt{\frac{2i\pi\hbar\epsilon}{m}}\left[\left(1 - \frac{i\epsilon}{\hbar}V(x)\right)\psi + \frac{i\hbar\epsilon}{2m}\frac{\partial^2\psi}{\partial x^2} + ...\right] \tag{5}$$

Something that I have (very purposefully) glazed over is the fact that the whole coefficient of this bracketed function, $A\sqrt{2i\pi\hbar\epsilon/m}$, *seems* to be of order $\epsilon^{1/2}$. However, as we are about to see, it actually isn't dependent on ϵ at all, which is why I didn't factor it into my above analysis.

Plugging the Taylor expansion of $\psi(x, t+\epsilon)$ into the left-hand-side of (3), and the above solution for $\mathcal{I}$ into the right-hand-side, we find that:

$$\psi + \epsilon\frac{\partial\psi}{\partial t} + ... = A\sqrt{\frac{2i\pi\hbar\epsilon}{m}}\left[\left(1 - \frac{i\epsilon}{\hbar}V(x)\right)\psi + \frac{i\hbar\epsilon}{2m}\frac{\partial^2\psi}{\partial x^2} + ...\right]$$

Before doing anything, we should notice that in the limit of $\epsilon \to 0$, the left-hand-side and right-hand-side should both be ψ, meaning that the factor A must be a function of ϵ such that it removes the coefficient $\sqrt{2i\pi\hbar\epsilon/m}$; i.e. it must be that:

$$A(\epsilon) = \sqrt{\frac{m}{2i\pi\hbar\epsilon}}$$

resulting in the equation:

$$\psi + \epsilon\frac{\partial\psi}{\partial t} + ... = \left(1 - \frac{i\epsilon}{\hbar}V(x)\right)\psi + \frac{i\hbar\epsilon}{2m}\frac{\partial^2\psi}{\partial x^2} + ... \tag{6}$$

The equation that results from the ϵ^0 terms is just $\psi(x,t) = \psi(x,t)$, which is exactly what we should expect of a zeroth-order approximation. It's the equation that results from the ϵ^1 terms that gives is Schrödinger's equation:

$$\frac{\partial\psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{i}{\hbar}V(x)\psi$$

To put this in it's standard form, multiply both sides by $i\hbar$, yielding:

$$i\hbar\frac{\partial\psi}{\partial t}(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}(x,t) + V(x)\psi(x,t) \tag{7}$$

which is the typical form of the 1-dimensional Schrödinger equation.

3 Some Concluding Thoughts

Clearly, Feynman was onto something from the get-go. He goes onto say, in his 1965 Nobel Lecture, that Jehle furiously copied down his notes and exclaimed that this was really something. Feynman went onto to rigorously derive the proper path integral formalism of non-relativistic quantum mechanics for his PhD dissertation. While path integrals aren't very useful in quantum mechanics, as evidenced by the fact that they are almost never taught in any courses, they are essential to the study of quantum field theory, which is the relativistic formulation of regular quantum mechanics.

There are some rather-large issues to deal with when transitioning from a non-relativistic quantum mechanical path integral to a relativistic quantum field theoretic path integral. The essential issue is the fact that in non-relativistic quantum mechanics, you work in a single-particle view-point: there exists one particle, described by the wavefunction $\psi(\mathbf{x}, t)$, and the wavefunction's evolution can be governed by a path integral over all possible paths $\mathbf{x}(t)$ in 4-dimensional spacetime.

In field theory, however, it is *impossible* to work in a single-particle viewpoint. The energies are allowed to get arbitrarily large, and any energy $\geq 2mc^2$, where m is the mass of the particle you are working with, will necessarily allow for the possibility of pair production, changing your one particle into three. Energies $\geq 4mc^2$ will allow for two pair-productions, yielding five particles. And so on, to arbitrarily large energies and thus arbitrarily large numbers of particles.

The correction to the single particle wavefunction view is to describe the existence of some field $\phi(x)$, defined at all 4-dimensional positions x, which contains information about the number of particles at each spacetime position x (there exists a number operator $\hat{N}$ which will tell you the number like $\hat{N}\phi(x) = N(x)\phi(x)$). The problem is that *each* particle in your theory, potentially a particle at *each* spacetime point x, must be evaluated individually in a path integral. This means an infinite number of path integrals must be evaluated; this is commonly phrased as "a field has an infinite number of degrees of freedom." While these path integrals can absolutely be computed, and essential results in quantum field theories (especially gauge field theories) are computed via path integrals, these path integrals are much, much more difficult to solve because they are over an infinite number of integrating variables.

Without this very important first step by Feynman, however, we might not have any path integral formalism for quantum mechanics, and therefore no path integral formalism for quantum field theory. I'm especially grateful to Herbert Jehle for showing Feynman that paper by Dirac, which pointed Feynman to the way to construct the path integral in the first place.

A Gaussian Integrals

The basic Gaussian integral is:

$$I = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx$$

Someone figured out a very clever trick to computing these integrals, and "higher-order" integrals of $x^n e^{-\alpha x^2}$. First, let's "square" this integral, in the sense:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-\alpha x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-\alpha y^2} dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\alpha(x^2+y^2)} dxdy \tag{8}$$

We want to keep each integral in terms of independent variables since the solution to each factor of I must be independent of the other.

Here comes the trick: we simply switch from Cartesian to polar coordinates. $dxdy$, each from $-\infty$ to ∞, describes an identical plane as $dA = rdrd\theta$, with r from 0 to ∞ and θ from 0 to 2π. With this change of variable comes the re-definition of distance in the plane as $x^2 + y^2 = r^2$. Since the plane is infinite, there's no difference in the circular plane and the rectangular plane. Plugging this substitution into I^2, equation (8) becomes:

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\alpha r^2} rdrd\theta = 2\pi \int_0^{\infty} re^{-\alpha r^2} dr \tag{9}$$

This is why we want to make this change of variable: now we can solve the integral with simple u-substitution. Define:

$$u = \alpha r^2 \quad \Rightarrow \quad du = 2\alpha rdr$$

Then, equation (9) becomes:

$$I^2 = \frac{\pi}{\alpha}\int_0^{\infty} e^{-u} du = -\frac{\pi}{\alpha}\left[e^{-\infty} - e^0\right] = \frac{\pi}{\alpha}$$

So, the solution to the integral I, which is our Gaussian integral, is just the square-root of the solution to the integral I^2, which is:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \tag{10}$$

Computing higher-order Gaussian integrals uses an equally clever trick. First of all, notice that all "odd-ordered" Gaussian integrals are zero:

$$\int_{-\infty}^{\infty} x^{2n+1} e^{-\alpha x^2} dx = 0 \tag{11}$$

This is because the Gaussian factor, $e^{-\alpha x^2}$, is an even function. Since x is an odd function, any power x^n where n is odd is an odd function, and thus so is $x^{2n+1}e^{-\alpha x^2}$. The integral of any odd function over the whole number line is always zero.

To find the "even-ordered" Gaussian integrals, we first notice the following:

$$-\frac{d}{d\alpha} e^{-\alpha x^2} = x^2 e^{-\alpha x^2}$$

which is exactly the function we're trying to integrate (at least for the second-order integral). In general, we would find that:

$$(-1)^n \frac{d^n}{d\alpha^n} e^{-\alpha x^2} = x^{2n} e^{-\alpha x^2} \tag{12}$$

We can then take our simple Gaussian integral, the "zeroth-order" Gaussian integral, and extend it to higher (even) order by noticing:

$$\int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2} dx = (-1)^n \int_{-\infty}^{\infty} \frac{d^n}{d\alpha^n} e^{-\alpha x^2} dx = (-1)^n \frac{d^n}{d\alpha^n} \left[\int_{-\infty}^{\infty} e^{-\alpha x^2} x \right]$$

Recalling the solution to the zeroth-order Gaussian integral, equation (11), we see that all (even) higher-order Gaussian integrals can be obtained with the formula:

$$\int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2} dx = (-1)^n \frac{d^n}{d\alpha^n} \left(\sqrt{\frac{\pi}{\alpha}} \right) \tag{13}$$

One can look up a general solution to the derivative on the right-hand-side, but almost always, in practice, it's only worth knowing up to the second-order Gaussian integral. This is found by taking the first derivative in terms of α:

$$\frac{d}{d\alpha} \left(\sqrt{\frac{\pi}{\alpha}} \right) = \frac{1}{2} \left(\sqrt{\frac{\pi}{\alpha}} \right)^{-1/2} * (-1) \frac{\pi}{\alpha^2} = -\frac{1}{2\alpha} \sqrt{\frac{\pi}{\alpha}}$$

So, our second-order Gaussian integral is:

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{2\alpha} \sqrt{\frac{\pi}{\alpha}} \tag{14}$$