Quantum Teleportation

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1 A Bit About Entangled States

Consider two particles, particle 1 and particle 2. particle 1 will be described by state vectors $|\psi\rangle^{(1)}$ in some Hilbert spaces H_1, and particle 2 will be described by some vectors $|\psi\rangle^{(2)}$ in some Hilbert space H_2. To describe the two-particle system composed of particles 1 and 2 (for example, a hydrogen atom would be described by the two-state system of the electron and proton), we need to form the tensor product space $H_1 \otimes H_2$ in which the two-particle states $|\psi\rangle^{(1)} \otimes |\psi\rangle^{(2)}$ will live.

If our chosen basis of H_1 is the set $\{|e_i\rangle^{(1)}\}$ and our chosen basis of H_2 is the set $\{|f_j\rangle^{(2)}\}$, then the basis of $H_1 \otimes H_2$ would be $\{|e_i\rangle^{(1)} \otimes |f_j\rangle^{(2)}\}$, and the most general two particle state $|\Psi\rangle \in H_1 \otimes H_2$ would be:

$$|\Psi\rangle = \sum_{ij} \alpha_{ij} |e_i\rangle^{(1)} \otimes |f_j\rangle^{(2)} \tag{1}$$

Note that what it means to be "entangled" is for the two particles to be inseparable under-measurement; the measurement of particle 1 must influence the measurement of particle 2, and vice-versa. Consider the state, represented in the z-spin basis of each particle,

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle^{(1)} |+\rangle^{(1)} + |-\rangle^{(1)} |-\rangle^{(2)} \right) \tag{2}$$

If you were to make a measurement on particle 1 (of its z-spin), and found the value to be $+1$ (ignoring the units of $\hbar/2$), then you'd *have* to know that particle 2 would have a z-spin of $+1$ as well. These particles are entangled.

What if our general state $|\Psi\rangle$ could be represented a slightly different way, via a factorization:

$$|\Psi\rangle = \left(\sum_i a_i |e_i\rangle^{(1)} \right) \otimes \left(\sum_j b_j |f_j\rangle^{(2)} \right) \tag{3}$$

I'm not talking about a change of basis or anything like that; I'm talking about simple factorization from elementary algebra. Well, in this case, no matter what state you measure particle 1 to be in, you will always measure particle 2 to be in the superposition of states

$$\sum_j b_j |f_j\rangle^{(2)}$$

and vice-versa for measuring the state of particle 1. These states are, therefore, *not* entangled, because measuring one particle doesn't pull the other out of a superposition of states and fix it into a single state; that is, the measurement of one has no affect on the other.

For two-particle states represented in the z-spin basis which are definitely entangled, there is a convenient change of basis that is usually made to what is known as the Bell basis:

$$\begin{aligned}
|\Phi_0\rangle^{(12)} &= \frac{1}{\sqrt{2}}\left(|+\rangle^{(1)} \otimes |+\rangle^{(2)} + |-\rangle^{(1)} |-\rangle^{(2)}\right) \\
|\Phi_1\rangle^{(12)} &= \frac{1}{\sqrt{2}}\left(|+\rangle^{(1)} \otimes |-\rangle^{(2)} + |-\rangle^{(1)} |+\rangle^{(2)}\right) \\
|\Phi_2\rangle^{(12)} &= \frac{i}{\sqrt{2}}\left(|+\rangle^{(1)} \otimes |-\rangle^{(2)} - |-\rangle^{(1)} |+\rangle^{(2)}\right) \\
|\Phi_3\rangle^{(12)} &= \frac{1}{\sqrt{2}}\left(|+\rangle^{(1)} \otimes |+\rangle^{(2)} - |-\rangle^{(1)} |-\rangle^{(2)}\right)
\end{aligned} \tag{4}$$

We can also give the transformation from the spin-basis to the Bell basis:

$$\begin{aligned}
|+\rangle^{(1)} \otimes |+\rangle^{(2)} &= \frac{1}{\sqrt{2}}\left(|\Phi_0\rangle^{(12)} + |\Phi_3\rangle^{(12)}\right) \\
|-\rangle^{(1)} \otimes |-\rangle^{(2)} &= \frac{1}{\sqrt{2}}\left(|\Phi_0\rangle^{(12)} - |\Phi_3\rangle^{(12)}\right) \\
|+\rangle^{(1)} \otimes |-\rangle^{(2)} &= \frac{1}{\sqrt{2}}\left(|\Phi_1\rangle^{(12)} - i\,|\Phi_2\rangle^{(12)}\right) \\
|-\rangle^{(1)} \otimes |+\rangle^{(2)} &= \frac{1}{\sqrt{2}}\left(|\Phi_1\rangle^{(12)} + i\,|\Phi_2\rangle^{(12)}\right)
\end{aligned} \tag{5}$$

2 A Simple Quantum Teleportation Experiment

Suppose that Alice has a quantum state

$$|\psi\rangle^{(C)} = \alpha\,|+\rangle^{(C)} + \beta\,|-\rangle^{(C)} \tag{6}$$

which she wants to give to Bob. She cannot duplicate the state and give it to Bob because of the no-cloning theorem. She can't measure α (and subsequently know β), as she only has one copy of $|\psi\rangle_C$ and therefore can only make one measurement on it. In order to know α, which would tell Alice the probability of finding $|\psi\rangle_C$ in the spin-up state, she'd have to perform many, many measurements on it. But she'd destroy her only copy of $|\psi\rangle_C$ by measuring it once. There are other things she could attempt, and each of them has an issue, but let's just stipulate the point that she needs to be very clever to send over the state to Bob.

A solution to this problem was presented by Bennet, et al., in 1993, using a new process they termed quantum teleportation. By utilizing an entangled pair of particles shared by Alice and Bob, Alice would very cleverly be able to send over her state to Bob in a way reminiscent of teleportation in science fiction, in the sense that once the state is in Bob's possession, it is no longer in Alice's possession.

Imagine that Alice and Bob have a method of producing an entangled pair of particles, in the Bell state:

$$|\Phi_0\rangle^{(AB)} = \frac{1}{\sqrt{2}}\left(|+\rangle^{(A)} \otimes |+\rangle^{(B)} + |-\rangle^{(A)} \otimes |-\rangle^{(B)}\right) \tag{7}$$

Alice would receive the particle labeled A and Bob would receive the particle labeled B, and they would each have some apparatus to perform measurements on their particles. I'll explain how they each chose to set up their apparatuses in a moment.

With the quantum state Alice wants to teleport to Bob, $|\psi\rangle_C$, the three particle state of this experiment will be:

$$|\Psi\rangle = |\Phi_0\rangle^{(AB)} \otimes |\psi\rangle^{(C)} \tag{8}$$

Expanding each state and taking the tensor product, we see:

$$\begin{aligned}|\Psi\rangle &= \left[\frac{1}{\sqrt{2}}\left(|+\rangle^{(A)} \otimes |+\rangle^{(B)} + |-\rangle^{(A)} \otimes |-\rangle^{(B)}\right)\right] \otimes \left[\alpha\,|+\rangle^{(C)} + \beta\,|-\rangle^{(C)}\right] \\ &= \frac{1}{\sqrt{2}}\Big(\alpha\,|+\rangle^{(A)} \otimes |+\rangle^{(B)} \otimes |+\rangle^{(C)} + \beta\,|+\rangle^{(A)} \otimes |+\rangle^{(B)} \otimes |-\rangle^{(C)} \\ &\qquad + \alpha\,|-\rangle^{(A)} \otimes |-\rangle^{(B)} \otimes |+\rangle^{(C)} + \beta\,|-\rangle^{(A)} \otimes |-\rangle^{(B)} \otimes |-\rangle^{(C)}\Big)\end{aligned}$$

The tensor product is commutative, so we can group the two particles that Alice can make measurements on, A and C, together:

$$\begin{aligned}|\Psi\rangle = \frac{1}{\sqrt{2}}\Big[&\left(|+\rangle^{(A)} \otimes |+\rangle^{(C)}\right) \otimes \alpha\,|+\rangle^{(B)} + \left(|+\rangle^{(A)} \otimes |-\rangle^{(C)}\right) \otimes \beta\,|+\rangle^{(B)} \\ &+ \left(|-\rangle^{(A)} \otimes |+\rangle^{(C)}\right) \otimes \alpha\,|-\rangle^{(B)} + \left(|-\rangle^{(A)} \otimes |-\rangle^{(C)}\right) \otimes \beta\,|-\rangle^{(B)}\Big]\end{aligned} \tag{9}$$

Note that I've also moved the constants α and β around within the tensor product, which is also allowed.

Now I am free to re-write the two-particle AC states in terms of the Bell bases; I just need to make sure that when Alice makes a measurement on her particles, her apparatus is set up to measure in the Bell basis. So, the three-particle state becomes:

$$\begin{aligned}|\Psi\rangle = \frac{1}{\sqrt{2}}\Big[&\frac{1}{\sqrt{2}}\left(|\Phi_0\rangle^{(AC)} + |\Phi_3\rangle^{(AC)}\right) \otimes \alpha\,|+\rangle^{(B)} + \frac{1}{\sqrt{2}}\left(|\Phi_1\rangle^{(AC)} - i\,|\Phi_2\rangle^{(AC)}\right) \otimes \beta\,|+\rangle^{(B)} \\ &+ \frac{1}{\sqrt{2}}\left(|\Phi_1\rangle^{(AC)} + i\,|\Phi_2\rangle^{(AC)}\right) \otimes \alpha\,|-\rangle^{(B)} + \frac{1}{\sqrt{2}}\left(|\Phi_0\rangle^{(AC)} - |\Phi_3\rangle^{(AC)}\right) \otimes \beta\,|-\rangle^{(B)}\Big]\end{aligned}$$

After this change of basis, we can group by Bell states:

$$\begin{aligned}|\Psi\rangle = \frac{1}{2}\Big[&|\Phi_0\rangle^{(AC)}\left(\alpha\,|+\rangle^{(B)} + \beta\,|-\rangle^{(B)}\right) + |\Phi_1\rangle^{(AC)}\left(\alpha\,|-\rangle^{(B)} + \beta\,|+\rangle^{(B)}\right) \\ &+ |\Phi_2\rangle^{(AC)}\left(i\alpha\,|-\rangle^{(B)} - i\beta\,|+\rangle^{(B)}\right) + |\Phi_3\rangle^{(AC)}\left(\alpha\,|+\rangle^{(B)} - \beta\,|-\rangle^{(B)}\right)\Big]\end{aligned} \tag{10}$$

Look at the first term in the above equation: it's identical to Alice's state that she wished to transport to Bob! So, if Alice were to measure her two-particle state and found it to be $|\Phi_0\rangle$, then her initial state would have successfully been teleported to Bob. We'll call the teleported state $|\psi\rangle^{(B)}$.

Notice, though, that this only works if Alice happens to measure $|\Phi_0\rangle$. But here's the brilliant part of the solution: Bob can design his measuring apparatus so that he receives Alice's state 100% of the time. In order to see this, we have to notice something about the other 3 possible states that Bob can measure. I'm going to write them in a peculiar way, but it will make the result clear:

$$\begin{aligned}
&(\text{if Alice measures } |\Phi_1\rangle) \quad \alpha\,|-\rangle^{(B)} + \beta\,|+\rangle^{(B)} \\
&(\text{if Alice measures } |\Phi_2\rangle) \quad \alpha\left(i\,|-\rangle^{(B)}\right) + \beta\left(-i\,|+\rangle^{(B)}\right) \\
&(\text{if Alice measures } |\Phi_3\rangle) \quad \alpha\,|+\rangle^{(B)} + \beta\left(-\,|-\rangle^{(B)}\right)
\end{aligned}$$

Each of these states that Bob's particle will have to be is actually $|\psi\rangle^{(B)}$ acted upon by one of the Pauli matrices (ignoring the factors of $\hbar/2$). For instance, the first state:

$$\alpha\,|-\rangle^{(B)} + \beta\,|+\rangle^{(B)} = \alpha\boldsymbol{\sigma}_1\,|+\rangle^{(B)} + \beta\boldsymbol{\sigma}_1\,|-\rangle^{(B)} = \boldsymbol{\sigma}_1\,|\psi\rangle^{(B)}$$

The second and third states are just $|\psi\rangle^{(B)}$ acted upon by $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$, respectively.

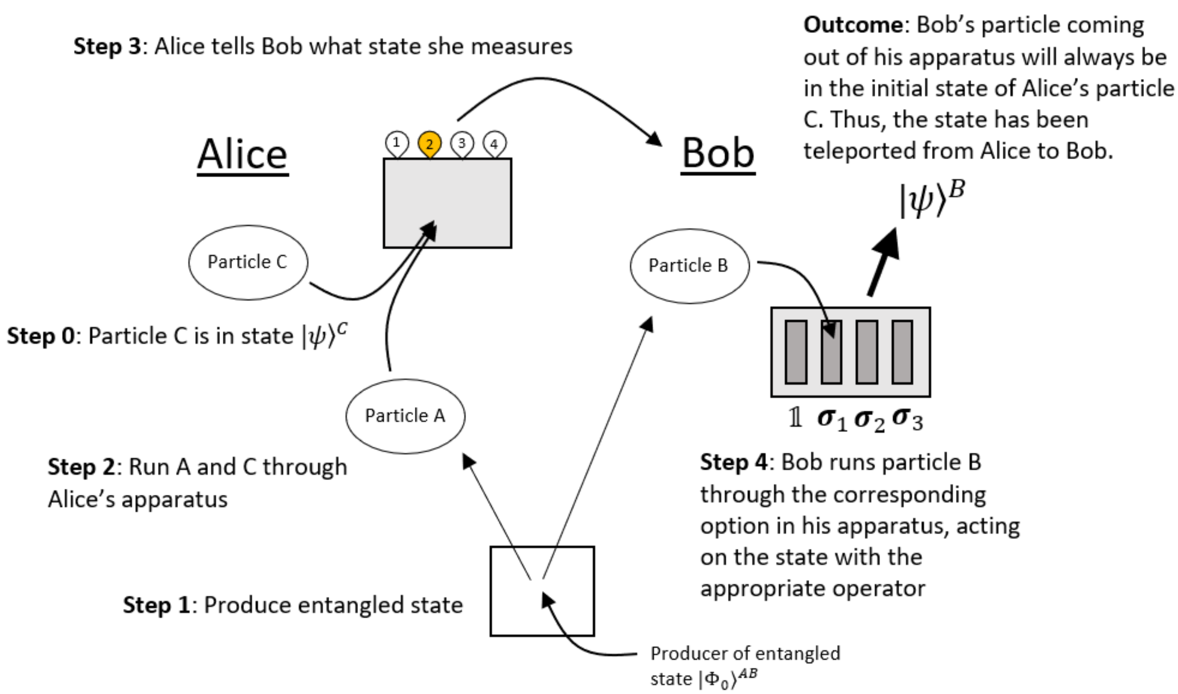
So, we see that the three-particle state is actually just

$$\begin{aligned}
|\Psi\rangle = \frac{1}{2}\Big(&|\Phi_0\rangle^{(AC)} \otimes |\psi\rangle^{(B)} + |\Phi_1\rangle^{(AC)} \otimes \boldsymbol{\sigma}_1\,|\psi\rangle^{(B)} + |\Phi_2\rangle^{(AC)} \otimes \boldsymbol{\sigma}_2\,|\psi\rangle^{(B)} \\
&+ |\Phi_3\rangle^{(AC)} \otimes \boldsymbol{\sigma}_3\,|\psi\rangle^{(B)}\Big)
\end{aligned} \tag{11}$$

Here's where Bob's apparatus comes in. Since for all Pauli matrices, $\boldsymbol{\sigma}_i^2 = 1$, if Bob's apparatus is built to hit a state such as $\boldsymbol{\sigma}_2\,|\psi\rangle^{(B)}$ with another $\boldsymbol{\sigma}_2$, then his resulting state would be

$$\boldsymbol{\sigma}_2\left(\boldsymbol{\sigma}_2\,|\psi\rangle^{(B)}\right) = (\boldsymbol{\sigma}_2)^2\,|\psi\rangle^{(B)} = |\psi\rangle^{(B)}$$

So, here's the complete experiment: Alice's apparatus will be set up so that when she runs her two-particle state AC through it, a light will indicate to her which bell state it's in, $|\Phi_\mu\rangle$, where $\mu = 0$ through 3. Alice will then transmit to Bob a *classical* signal, i.e. a telephone call, that let's him know which light lit up on her apparatus. Bob will then run his state B, which is still entangled with Alice's state, through his own apparatus, which has options 0 through 3. Option 0 does nothing to the state, and option i ($i = 1$ through 3) operators on the state a $\boldsymbol{\sigma}_i$. This way, no matter what the outcome of Alice's measurement, 100% of the time, Bob's state is $|\psi\rangle^{(B)}$, and Alice's quantum state has been successfully teleported to Bob. One such experiment is outlined in the following figure.



Possible Quantum Teleportation Experiment