Tunneling Through a Potential Barrier

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1 Introduction

Consider a classical object moving along the ground with some kinetic energy K, when it approaches a hill. The height of the hill is such that the gravitational potential energy of the object at the top of the hill would be V. If $K > V$, the object will climb over the hill and keep moving, but if $K < V$, the object won't have enough kinetic energy to reach the top of the hill, and will turn around and move in the opposite direction once it reached a height such that $K \to 0$. This situation is the classical analogy to a free quantum particle, with some total energy E, approaching a potential barrier of some height V. The *expected* result, from a classical analysis, is that the object would pass through the barrier if $E > V$, but be turned around by the barrier if $E < V$. However, this isn't the case: there's a non-zero probability that the particle will *still* pass through the barrier if $E < V$. This is known as quantum tunneling, or simply tunneling.

Tunneling has an actual, observable impact on life. It most commonly is seen in α-decay of radioactive nuclei. An alpha particle, bound in a nucleus, often never has enough energy to overcome the strong force binding it. However, it has a change to simply tunnel through the potential barrier and pop out on the other side a free α particle. This is exactly how this decay process works. This explanation was proposed by George Gamow in 1928[1], and we will cover it at the end of this note.

Tunneling also puts limits on how efficient microprocessors can become. As it currently stands, the transistors used in computer processors are the so-called "field-effect" transistors (FETs), which use a controllable electric field (in an "on-off" configuration) to either prevent electrons from moving through the transistor, or to allow them to move through. The problem is that they require tens of thousands of electrons to produce a signal currently. The ideal scenario would be to produce a single-electron transistor, which would only require a single electron to move across the transistor to produce a signal. However, counting on a single electron to sit inside a transistor, holding it back with an electric potential barrier, isn't going to work 100% of the time, since the electron can tunnel through this barrier. It's very different in FETs, which require tens of thousands of electrons to produce a signal, so a single electron tunneling won't produce a signal, and the odds of ten thousand tunneling at the same time is essentially zero.

2 Direct Solution of Schrödinger's Equation

Consider a free particle with energy $E > 0$ encountering a potential barrier of height $V_0 > E$, which extends from $x = 0$ to $x = a$. This is not a *classically* passable barrier, since the particle doesn't have enough energy to overcome the potential. But we will show that there does exist a non-zero

[1] G. Gamow, "Zur Quantentheorie des Atomkernes (On the quantum theory of the atomic nucleus)," *Z. Phys*, **51**, 204 (1928).

probability of the particle doing exactly that: passing through the barrier. To begin, let's divide the x-axis into three regions:

$$\begin{aligned} &\text{Region I:} \quad -\infty < x < 0 \\ &\text{Region II:} \quad 0 < x < a \\ &\text{Region III:} \quad a < x < \infty \end{aligned}$$

In Region I and III, Schrödinger's equation is going to be:

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi \quad (k \in \mathbb{R}) \tag{1}$$

which is going to have the solutions:

$$\begin{aligned} \text{(Region I)} \quad \psi_I(x) &= Ae^{-ikx} + Be^{ikx} \\ \text{(Region III)} \quad \psi_{III}(x) &= Fe^{-ikx} \end{aligned} \tag{2}$$

Imagine that the problem was as depicted in Figure 1, where I am using the convention that a wave e^{-ikx} moves in the $+x$-direction. In the figure, it's pointed out that a wave moving in the $-x$-direction in region III is unphysical: consider the right-moving wave in Region I as the wave indident on the boundary, and the left-moving wave in Region I as the reflected wave, then a right-moving wave in region III is clearly a transmitted wave, but there is no physical interpretation for a left-moving wave in region III.

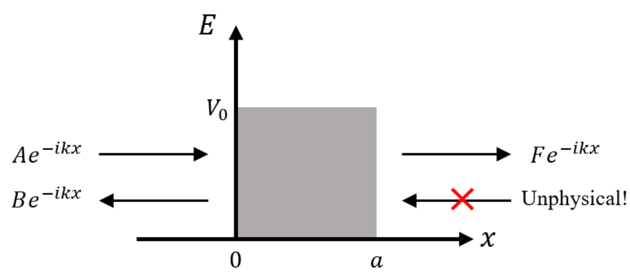

Figure 1: The physical solutions to Schrödinger's equation for $V_0 > E$.

The transmission coefficient, the probability of passing through the barrier, is just going to be the relative probability of measuring a right-moving wave in region III vs. a right-moving wave in region I, i.e.:

$$T = \frac{|F|^2}{|A|^2} \tag{3}$$

while the reflection coefficient, the probability of bouncing off of the barrier, is going to be the relative probability of a left-moving vs. right-moving wave in region I, i.e.:

$$R = \frac{|B|^2}{|A|^2} \tag{4}$$

Note that in neither of the above scenarios is $|A| = 1$; this is because you can't normalize free particle solutions. A relation between transmission and reflection coefficients naturally exists due to their probabilistic interpretations:

$$R + T = 1 \tag{5}$$

though there exists the same relation in classical optics without the probabilistic interpretation.

Now, in Region II, Schrödinger's equation is going to be:

$$\frac{d^2\psi}{dx^2} = -\frac{2m(E-V_0)}{\hbar^2}\psi = +\kappa^2\psi \quad (\kappa \in \mathbb{R}) \tag{6}$$

where the inclusion of the negative sign in the definition of κ^2 is necessary to make κ real since $E - V_0 < 0$. The wavefunction in Region II is therefore:

$$\psi_{II}(x) = Ce^{\kappa x} + De^{-\kappa x} \tag{7}$$

The boundary conditions are going to be that the wavefunction and its derivative are continuous at $x = 0$ and $x = a$. This gives us four equations, though there are 5 unknowns: A, B, C, D, and F. However, recall that the reflection and transmission coefficients are defined as ratios with respect to $|A|^2$, so our four boundary conditions can give the four coefficients B, C, D, and E as ratios with respect to A, allowing us to find R and T. These four equations are:

$$\begin{aligned}
& \psi_I(0) = \psi_{II}(0) & \\
\Rightarrow\quad & A + B = C + D & \text{(BCI)} \\
& \psi_I'(0) = \psi_{II}'(0) & \\
\Rightarrow\quad & -ik\,(A - B) = \kappa\,(C - D) & \text{(BCII)} \\
& \psi_{III}(a) = \psi_{II}(a) & \\
\Rightarrow\quad & Fe^{-ika} = Ce^{\kappa a} + De^{-\kappa a} & \text{(BCIII)} \\
& \psi_{III}'(a) = \psi_{II}'(a) & \\
\Rightarrow\quad & -ikFe^{-ika} = \kappa\,(Ce^{\kappa a} - De^{-\kappa a}) & \text{(BCIV)}
\end{aligned}$$

We can use (BCIII) and (BCIV) to solve for C and D as functions of F. First, we want to divide (BCIV) by κ, then multiply (BCIII) and (BCIV) by $e^{\kappa a}$, yielding:

$$Fe^{-ika}e^{\kappa a} = Ce^{2\kappa a} + D$$

$$-\frac{ik}{\kappa}Fe^{-ika}e^{\kappa a} = Ce^{2\kappa a} - D$$

Adding these equations, and solving for C, gives:

$$C = \frac{1}{2}\left(1 - \frac{ik}{\kappa}\right)Fe^{-ika}e^{\kappa a} \tag{8}$$

We need to use (BCIII) and (BCIV) again to solve for D, doing basically the opposite of before. We'll divide (BCIV) by $-\kappa$ this time, and then multiply (BCIII) and (BCIV) by $e^{-\kappa a}$, resulting in:

$$Fe^{-ika}e^{-\kappa a} = C + De^{-2\kappa a}$$

$$\frac{ik}{\kappa}Fe^{-ika}e^{-\kappa a} = -C + De^{-2\kappa a}$$

And adding these equations, and solving for D, yields:

$$D = \frac{1}{2}\left(1 + \frac{ik}{\kappa}\right)Fe^{-ika}e^{-\kappa a} \tag{9}$$

Now we want to take these results for C and D and plug them into (BCI) and (BCII). First, let's work with (BCI):

$$\begin{aligned} A+B &= \frac{1}{2}\left(1-\frac{ik}{\kappa}\right)Fe^{-ika}e^{\kappa a}+\left(1+\frac{ik}{\kappa}\right)Fe^{-ika}e^{-\kappa a} \\ &= \frac{1}{2}(e^{\kappa a}+e^{-\kappa a})Fe^{-ika}+\frac{1}{2}\frac{ik}{\kappa}(e^{-\kappa a}-e^{\kappa a})Fe^{-ika} \\ &= \left[\cosh(\kappa a)-\frac{ik}{\kappa}\sinh(\kappa a)\right]Fe^{-ika} \qquad (EI) \end{aligned}$$

For (BCII), first we'll divide both sides by κ, and then plug in the results for C and D. We'll get a very similar result as above:

$$\begin{aligned} -\frac{ik}{\kappa}(A-B) &= \frac{1}{2}\left(1-\frac{ik}{\kappa}\right)Fe^{-ika}e^{\kappa a}-\frac{1}{2}\left(1+\frac{ik}{\kappa}\right)Fe^{-ika}e^{-\kappa a} \\ &= \frac{1}{2}(e^{\kappa a}-e^{-\kappa a})Fe^{-ika}-\frac{1}{2}\frac{ik}{\kappa}(e^{\kappa}+e^{-\kappa a})Fe^{-ika} \\ &= \left[\sinh(\kappa a)-\frac{ik}{\kappa}\cosh(\kappa a)\right]Fe^{-ika} \end{aligned}$$

Now moving the factor of $-ik/\kappa$ from the left-side to the right-side, we have:

$$A-B=\frac{i\kappa}{k}\left[\sinh(\kappa a)-\frac{ik}{\kappa}\cosh(\kappa a)\right]Fe^{-ika} \qquad (EII)$$

Adding these equations (EI) + (EII) yields a solution for A:

$$2A=\left[2\cosh(\kappa a)+i\left(\frac{\kappa}{k}-\frac{k}{\kappa}\right)\sinh(\kappa a)\right]Fe^{-ika}$$

So, we can solve for F as a function of A, exactly as we wanted to when we started:

$$F=\frac{2e^{-ika}}{2\cosh(\kappa a)+i\left(\frac{\kappa}{k}-\frac{k}{\kappa}\right)\sinh(\kappa a)}A$$

Now we need to take the modulus-square of both sides, and solve for $T=|F|^2/|A|^2$. Notice something, though: the modulus square is going to involve a $e^{-ika}*e^{ika}=1$ in the numerator, and a $(A+iB)(A-iB)=A^2+B^2$ in the denominator. Because the numerator will simply be a number, but the denominator will be a more complicated function, I'll express the transmission coefficient as an inverse T^{-1} instead, for convenience. Thus, we get:

$$T^{-1}=\cosh^2(\kappa a)+\frac{1}{4}\left(\frac{\kappa}{k}-\frac{k}{\kappa}\right)^2\sinh^2(\kappa a)$$

There's one last bit of simplification we need to perform. Recalling that $\cosh^2(x)=1+\sinh^2(x)$ for hyperbolic trig functions, we see T^{-1} becomes:

$$T^{-1}=1+\frac{1}{4}\left[4+\left(\frac{\kappa}{k}-\frac{k}{\kappa}\right)^2\right]\sinh^2(\kappa a)=1+\frac{1}{4}\left[\frac{4k^2\kappa^2+\kappa^4+k^4-2k^2\kappa^2}{k^2\kappa^2}\right]\sinh^2(\kappa a)$$

Noting that $\kappa^4 + k^2 + 2k^2\kappa^2 = (\kappa^2 + k^2)^2$, we find one of our final solutions for T^{-1}:

$$T^{-1} = 1 + \frac{1}{4}\left(\frac{\kappa}{k} + \frac{k}{\kappa}\right)^2 \sinh^2(\kappa a) \tag{10}$$

This isn't the only way to express the transmission coefficient, though. Often times, it's preferred that it be expressed in terms of the energies E and V_0 that set the details of the problem. Recalling our definitions of k and κ, from equations (1) and (6), we see that:

$$\frac{\kappa}{k} + \frac{k}{\kappa} = \frac{\kappa^2 + k^2}{k\kappa} = \frac{(V_0 - E) + (E)}{\sqrt{E(V_0 - E)}} = \frac{V_0}{\sqrt{E(V_0 - E)}}$$

So, a popular alternate to T^{-1} given above is:

$$T^{-1} = 1 + \left(\frac{V_0^2}{4E(V_0 - E)}\right) \sinh^2(\kappa a) \tag{11}$$

The above solution, plotted against E/V_0, is given in Figure 2 below.

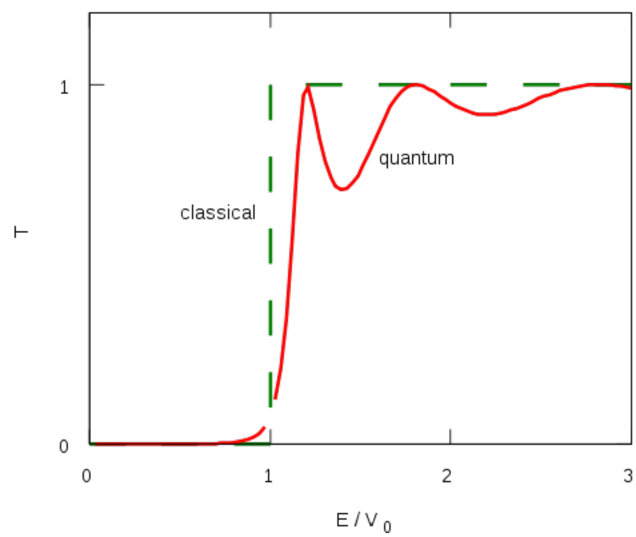

Figure 2: T for tunneling. *Image credit: WikimediaCommons, author Bamse.*

3 Approximate Solutions for Tunneling

First, let's look at the "weak tunneling" limit, in which $\kappa a \gg 1$. To see how T^{-1} should act, let's re-write $\sinh(\kappa a)$ in terms of exponentials:

$$\sinh(\kappa a) = \frac{1}{2}(e^{\kappa a} - e^{-\kappa a})$$

What's going to happen in the $\kappa a \gg 1$ limit? Well, $e^{\kappa a}$ is going to blow up, but $e^{-\kappa a}$ is going to drop to zero. We don't want our result to blow up, but keep in mind we're not analyzing the transmission coefficient, but the *inverse* of it. This means in order to keep T from blowing up, we need to drop the $e^{-\kappa a}$ and only keep the $e^{-\kappa a}$. Even though T^{-1} will blow up, T will remain finite. So, we have:

$$T^{-1} \approx 1 + \left(\frac{\kappa}{k} + \frac{1}{16}\frac{k}{\kappa}\right)^2 e^{2\kappa a} \approx \frac{1}{16}\left(\frac{\kappa}{k} + \frac{k}{\kappa}\right)^2 e^{2\kappa a}$$

where the second approximation is because, as $e^{2\kappa a}$ blows up, it gets much larger than 1, so we can ignore it. So, in the limit $\kappa a \gg 1$, the transmission coefficient is:

$$T = \frac{16k^2\kappa^2}{(\kappa^2 + k^2)^2} e^{-2\kappa a} \tag{12}$$

Or, expressed in terms of E and V_0, the transmission coefficient is:

$$T = \frac{16E(V_0 - E)}{V_0^2} e^{-2\kappa a} \tag{13}$$

Tunneling isn't typically analyzed the way I've done it so far. Typically, tunneling is looked at using what's known as the WKB approximation[2]. The WKB approximation is used when a potential barrier encountered is constant or nearly constant. This means that we can, of course, use the WKB approximation to good effect for tunneling, since the potential barrier is constant during the process. The approximation that's actually made in WKB is to assume that if $V(x)$ varies slowly, relative to some wavelength λ (i.e. the de Broglie wavelength of the particle), then we assume a solution:

$$\psi(x) = A(x)e^{i\phi(x)} \tag{14}$$

with $\phi(x)$ either real or imaginary, depending on how the energy compares to the potential barrier at x. Since we're treating $V(x)$ as varying slowly, we also want to treat $A(x)$ as varying slowly, such that we ignore all derivatives $A^{(n)}(x)$ for $n > 1$, i.e. we only consider the first derivative of $A(x)$.

Taking the first derivative of ψ, we have:

$$\frac{d\psi}{dx} = \left(\frac{dA}{dx} + iA\frac{d\phi}{dx}\right) e^{i\phi(x)}$$

So, the second derivative of ψ, which we will plug into Schrödinger's equation, will be:

$$\frac{d^2\psi}{dx^2} = \left(\frac{d^2A}{dx^2} + 2i\frac{dA}{dx}\frac{d\phi}{dx} + iA\frac{d^2\phi}{dx^2} - A\left(\frac{d\phi}{dx}\right)^2\right) e^{i\phi(x)} \tag{15}$$

If we re-write Schrödinger's equation such that $d^2\psi/dx^2$ sits by itself, we can define the function $p(x)$ such that:

$$\frac{d^2\psi}{dx^2} = -\frac{2m(E - V(x))}{\hbar^2}\psi = -\frac{p(x)^2}{\hbar^2}\psi \tag{16}$$

Setting equation (15) equal to $-p^2/\hbar^2$, we have:

$$\frac{d^2A}{dx^2} + 2i\frac{dA}{dx}\frac{d\phi}{dx} + iA\frac{d^2\phi}{dx^2} - A\left(\frac{d\phi}{dx}\right)^2 = -\frac{p^2}{\hbar^2}A$$

The above equation can be split into a real part:

$$\frac{d^2A}{dx^2} = A\left[-\frac{p^2}{\hbar^2} + \left(\frac{d\phi}{dx}\right)^2\right] \tag{17}$$

[2] Named after Gregor Wentzel, Hans Kramers, and Léon Brillouin.

and an imaginary part:

$$2\frac{dA}{dx}\frac{d\phi}{dx} + A\frac{d^2\phi}{dx^2} = \frac{d}{dx}\left(A^2\frac{d\phi}{dx}\right) = 0 \tag{18}$$

The solution to the imaginary part is just that $A^2\phi'$ is equal to a constant, which we can call C^2, meaning:

$$A = \frac{C}{\sqrt{\phi'}} \tag{19}$$

Recall that the WKB approximation requires us to set all derivatives of A of order-2 or higher equal to zero. This means that the real part of Schrödinger's equation reduces to:

$$\left(\frac{d\phi}{dx}\right) = \frac{p^2}{\hbar^2} \quad \Rightarrow \quad \frac{d\phi}{dx} = \pm\frac{p(x)}{\hbar}$$

Solving this simple differential equation for $\phi(x)$, we have:

$$\phi(x) = \pm\frac{1}{\hbar}\int p(x)dx \tag{20}$$

We know have everything we need to find the wavefunction. Note that $\sqrt{\phi'} = \sqrt{p(x)/\hbar}$, but we can just absorb $\hbar$ into the constant C. Thus, we have:

$$\psi(x) \approx \frac{C}{\sqrt{p(x)}}e^{\pm\frac{1}{\hbar}\int p(x)dx} \tag{21}$$

Notice that I left the integral for $\phi(x)$ as an indefinite integral. This is fine because any integration constant will appear in ψ as a $e^{C'}$, which can just be absorbed into C and ignored. However, since $p(x)$ isn't strictly real (it depends on whether $E > V(x)$ or $E < V(x)$ at x), absorbing this constant into C will make C a potentially imaginary number.

The general solution for $\psi(x)$ will, of course, be a linear combination of the two solutions found above:

$$\psi(x) \approx \frac{C}{\sqrt{p(x)}}e^{\frac{1}{\hbar}\int p(x)dx} + \frac{C'}{\sqrt{p(x)}}e^{-\frac{1}{\hbar}\int p(x)dx} \tag{22}$$

Notice, also, that the modulus-square of $\psi(x)$ is given by:

$$|\psi(x)|^2 \approx \frac{|C|^2}{p(x)} \tag{23}$$

so the probability of finding the particle at x is inversely proportional to the classical momentum p at x. This should be obvious: if a particle is moving very fast, i.e. has a large p, then it won't be very likely to be measured in a location, since it moves on from a location very rapidly.

For tunneling, we want to consider the case where E is strictly less than $V(x)$ throughout the potential barrier, i.e. from $0 \leq x \leq a$. This means that C will be imaginary, and everything must be calculated with the modulus of $p(x)$:

$$\psi(x) \approx \frac{C}{\sqrt{|p(x)|}}e^{\frac{1}{\hbar}\int_0^x |p(x)|dx} + \frac{D}{\sqrt{|p(x)|}}e^{-\frac{1}{\hbar}\int_0^x |p(x)|dx} \tag{24}$$

where I am using the coefficients in the same manner as in equation (6). Outside of the potential barrier, we have the typical free particle solutions given by (2).

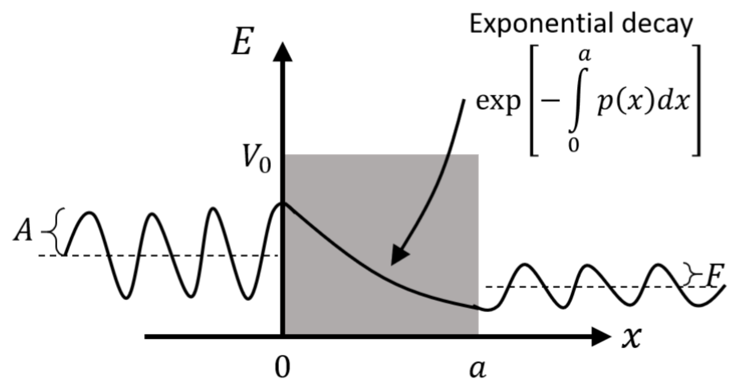

Figure 3: Wavefunction vs. position for a tunneling particle.

While we could go through the same procedure as in the previous section, setting the wavefunctions and their derivatives equal at $x = 0$ and $x = a$, we can actually guess at the solution based solely on looking at the amplitude throughout particle's motion from $-\infty$ to $+\infty$. The wavefunction vs. position is plotted in Figure 3.

As clear in the figure, the particle reaches the potential with an amplitude of A, which then undergoes an expoential decay, becoming F after it exits the barrier. So, F should be related to A like:

$$F \approx Ae^{-\frac{1}{\hbar}\int_0^a |p(x)|dx}$$

where this is an approximate solution not because we're guessing as to what it is, but because the amount of amplitude decay is approximate. So, the transmission coefficient, which is just $|F|^2/|A|^2$, is:

$$T \approx \exp\left(-\frac{2}{\hbar}\int_0^a \sqrt{2m[V(x) - E]}dx\right) \tag{25}$$

where I substituted $|p(x)|$ into the solution as it was defined, but reversing the order of subtraction $E - V(x) \to V(x) - E$ to make $|p(x)|$ a real number.

Let's compare the WKB result with our weak limit result. All of our WKB analysis was for a general, potentially varying $V(x)$. But the case of tunneling that we are considering, $V(x) = V_0$ is a constant. So, the WKB result is that:

$$T \approx \exp\left(-2\sqrt{\frac{2m(V_0 - E)}{\hbar^2}}a\right)$$

where I pulled the $\hbar$ into the square root as $\hbar^2$ to keep with the common convention. Looking back to equation (13) for the weak limit solution, and all the way back to (6) for the definition of κ, the transmission coefficient in the weak limit was found to be:

$$T \approx \frac{16E(E - V_0)}{V_0^2}\exp\left[-2\sqrt{\frac{2m(V_0 - E)}{\hbar^2}}a\right]$$

Clearly, both the WKB approximation and the weak limit give the same exponential decay of the transmission coefficient, but the weak limit solution is going to be more accurate in the case of a *constant* potential barrier, since it was derived as the asymptotic behavior of the exact solution. However, the WKB approximation is going to, obviously, be the superior solution for a non-constant potential barrier, as that's what it was meant to be used for.

4 Gamow Theory of Alpha Decay

In Gamow's theory of α-decay, he modeled the potential energy of the α-particle, over distance, as being some constant, negative value $-U$ within t the nucleus, extending from 0 to some R, and then the potential energy being Coulomb for $r > R$. This is plotted in Figure 4 below.

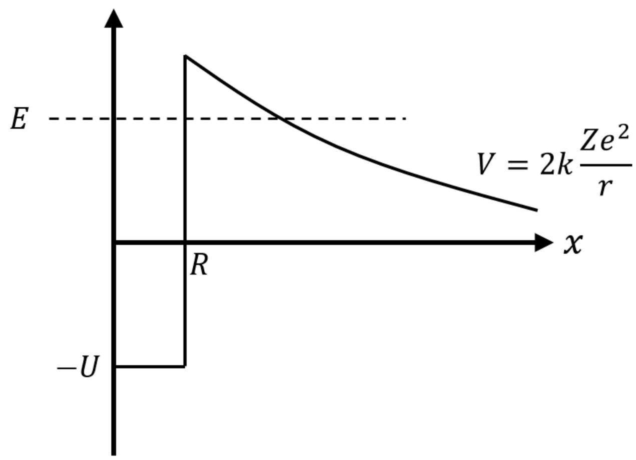

Figure 4: Potential barrier for an α particle of energy E.

If the α-decay occurs in a nucleus with an atomic number $Z+2$, then the Coulomb barrier is:

$$V(r) = 2k\frac{Ze^2}{r} \tag{26}$$

Since this is clearly varying with position, we need to use our WKB result. If we define r_2 to be the value of r when it exits the Coulomb barrier, i.e. the point at which $V(r_2) = E$, then r_2 will be given by:

$$r_2 = 2k\frac{Ze^2}{E} \tag{27}$$

Then, the transmission coefficient will be:

$$T \approx \exp\left[-\frac{2}{\hbar}\int_R^{r_2}\sqrt{2m\left(2k\frac{Ze^2}{r} - E\right)dr}\right]$$

Notice that we can say $2kZe^2 = Er_2$, so we can pull out a common factor of E from the square root, giving:

$$T \approx \exp\left[-\frac{2\sqrt{2mE}}{\hbar}\int_R^{r_2}\sqrt{\frac{r_2}{r} - 1dr}\right]$$

The solution to this integral should just be looked up:

$$T \approx \exp\left[-\frac{2\sqrt{2mE}}{\hbar}\left(r_2\cos^{-1}\sqrt{\frac{R}{r_2}} - \sqrt{R(r_2 - R)}\right)\right]$$

For a realistic decay scenario, E should be fairly far below the Coulomb barrier at R, thus $r_2 \gg R$. The arccosine term is dependent upon R/r_2, which tends to zero in the realistic case.

This means that $\theta \to \pi/2$, but remains slightly below it. If we define a very small angle ϵ such that $\theta = \pi/2 - \epsilon$, then we can use the angle-difference identity for cosine to show:

$$\cos\theta = \cos\left(\frac{\pi}{2} - \epsilon\right) = \cos\left(\frac{\pi}{2}\right)\cos\epsilon + \sin\left(\frac{\pi}{2}\right)\sin\epsilon \approx \epsilon$$

since the small-angle approximation for sine is $\sin\epsilon \approx \epsilon$. So, $\cos\theta = \sqrt{R/r_2}$, and thus $\epsilon = \sqrt{R/r_2}$, which means that the arccosine, which equals θ, is $\pi/2 - \epsilon$, or:

$$\cos^{-1}\sqrt{\frac{R}{r_2}} = \frac{\pi}{2} - \sqrt{\frac{R}{r_2}}$$

So, noting that $\sqrt{R(r_2 - R)} \approx \sqrt{r_2 R}$, the transmission coefficient is:

$$T \approx \exp\left[-\frac{2\sqrt{2mE}}{\hbar}\left(\frac{\pi}{2}r_2 - 2\sqrt{r_2 R}\right)\right] \tag{28}$$

It might make more sense to re-insert r_2 as a function of Z and E, so that we don't have to worry about compute r_2 for every case we come across. If we do so, we can define two constants, $\mathcal{E}$ and $\mathcal{R}$, such that:

$$\begin{aligned}\mathcal{E} &= 2ke^2\frac{\pi\sqrt{2m}}{\hbar} = 3.96\ \mathrm{MeV}^{1/2}\\ \mathcal{R} &= 8\sqrt{ke^2}\frac{\sqrt{m}}{\hbar} = 2.97\ \mathrm{fm}^{-1/2}\end{aligned} \tag{29}$$

such that the transmission coefficient becomes:

$$T \approx \exp\left[-\mathcal{E}\frac{Z}{\sqrt{E}} + \mathcal{R}\sqrt{ZR}\right] \tag{30}$$

We can use the semi-empirical mass formula (based on the liquid drop model of the atom, also proposed by George Gamow) to define the atomic radius:

$$R = r_0 A^{1/3} \tag{31}$$

where $r_0 \approx 1.25$ fm, and A is the nucleon number of the radioactive nucleus. A should *include* the 4 nucleons of the α particle, since the boundary of the nucleus is determined *prior* to the α particle beginning its tunneling, whereas we defined the proton number Z to be *excluding* the 2 protons in the α particle, since we used it to define the Coulomb repulsion between the left-over nucleus and the α particle. For instance, uranium-238, which is an α emitter, has 92 protons prior to decay. For this problem, we would set $A = 238$, since the total number of nucleons including the α particle defines R, but we would set $Z = 90$ to define the Coulomb barrier, since it is due to the interaction between the left-over nucleus (thorium-234) and the α particle.

The last thing to compute is the lifetime of an α emitting nucleus. If we consider an α particle as bouncing around on the inside of an atom, then it has to traverse, on average, the diameter of the atom between collisions, or $2R$. If it moves with a speed of v inside the nucleus, then the frequency of collisions is:

$$f = \frac{v}{2R}$$

Every time the α particle hits the Coulomb barrier, i.e. the boundary of the nucleus, it has a probability T of penetrating the barrier and escaping. So, the probability of emission will be $vT/2R$, and the lifetime τ will the inverse of this probability:

$$\tau \approx \frac{v}{2R} \exp\left(-\mathcal{E}\frac{Z}{\sqrt{E}} + \mathcal{R}\sqrt{ZR}\right) \tag{32}$$

There are many ways of estimating the speed v of an α particle in a nucleus. The simplest way is to use a non-relativistic approach and set $E = \frac{1}{2}mv^2$. The actual prediction of τ isn't going to be great, because there isn't really any particularly good way of estimating v. However, the behavior of $\ln\tau$ is dominated by the factor of $1/\sqrt{E}$, and this behavior agrees perfectly with experiment, as shown in Figure 5[3].

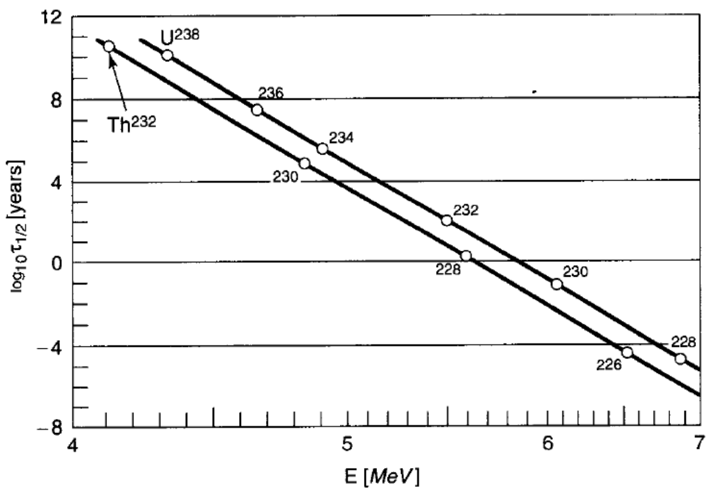

Figure 5: Theoretic vs. observed half-lives of α emitters. *Image credit: Griffiths.*

[3]Taken from D.J. Griffiths, "Introduction to Quantum Mechanics," *Prentice Hall* (1995).